GENUS 2 FIBRATIONS

by

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Preface

I would like to thank my supervisor Ben Moonen for his guidance in my thesis. His explanations have helped me develop geometric intuition, and get a better understanding of algebraic geometry. I would also like to thank Victoria Hoskins for accepting the role of second reader of this thesis.

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Introduction

In algebraic geometry, the theory of algebraic surfaces is a very classical topic, which has been studied intensively since the late 19th century. Nevertheless, it remains a difficult subject and there is still much unknown. For example unlike curves, there is no comprehensive classification of surfaces. Curves, on the other hand, are much better understood. Therefore, in order to understand a surface S better, we could search for a morphism $f\colon S \to B$ to a smooth curve, such that the fibres are connected and f is not constant. Such a morphism we will call a *fibration*, and it allows us to view S as a one-dimensional family of curves. This is used to great effect in the study of ruled and elliptic surfaces.

For a fibration almost all fibres are smooth of the same genus g, which we will call *the genus of the fibration*. In this thesis, the main focus will be the study of genus 2 fibrations, based on the approach taken by Catanese and Pignatelli in [8].

For a genus 2 fibration $f : S \to B$ we will introduce its relative canonical model $\mathfrak{X}$, which is a surface for which we have a canonical morphism $S \to \mathfrak{X}$ over B, which is an isomorphism on smooth fibres. While the fibres of f can be complicated, the fibres of $\mathfrak{X} \to B$ have a simple description: they are isomorphic to the homogeneous spectrum of $\mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6)$ with $\deg(x_0) = \deg(x_1) = 1$, $\deg(y) = 2$ and $\deg(z) = 3$ and Q_2, Q_6 polynomials of degree 2 and 6, respectively.

One goal of this thesis is to get some understanding of the singularities of $\mathfrak{X}$ for a genus 2 fibration. In order to do this, we construct a surface $\mathcal{C}$ with a double cover $\phi\colon \mathfrak{X} \to \mathcal{C}$, which is easier to understand: the fibres of $\mathcal{C} \to B$ have genus 0 and are easily described. We will describe the singularities of $\mathcal{C}$ and the ramification locus of ϕ, which together determines the singularities on $\mathfrak{X}$.

We construct this surface as follows. On smooth fibres of f, we can consider the hyperelliptic involution, and this induces an involution on S, which in turn induces an involution $\iota\colon \mathfrak{X} \to \mathfrak{X}$ on the relative canonical model. Using this involution we can consider the quotient $\mathcal{C} := \mathfrak{X}/\langle\iota\rangle$ and the double cover $\phi : \mathfrak{X} \to \mathcal{C}$. The surface $\mathcal{C}$ can be embedded in a $\mathbb{P}^2$-bundle $\mathbb{P}(\mathscr{R}_2)$ over B, and using this embedding we see the restriction of $\mathfrak{X} \to \mathbb{P}(\mathscr{R}_2)$ to a smooth fibre F is simply the bicanonical map $F \to \mathbb{P}^2$. This map on F is ramified at exactly 6 points, so we see the ramification divisor of ϕ consists of 6 branches, while the ramification locus consists of those branches combined with a finite set of isolated points on $\mathfrak{X}$.

Finally, using the understanding of the singularities of $\mathcal{C}$ and the ramification locus of ϕ, we describe those singularities in $\mathfrak{X}$ which do not lie on the singularities of the ramification divisor. This then used to see which types of fibres of $S \to B$ this gives rise to.

Chapter 1

Tools from Algebraic Geometry

In this chapter we collect some tools from algebraic geometry that will be needed in the next chapters.

1.1 Notation and conventions

1.1.1 Throughout this thesis, we work over the field $\mathbb{C}$ of complex numbers. We expect that many results discussed in this thesis will be valid more generally; however, as several results that we need are available in the literature only for complex varieties we will restrict to that case.

By a *variety* we mean an integral separated scheme of finite type over $\mathbb{C}$. Varieties of dimension 1 are called *curves*, while varieties of dimension 2 are called *surfaces*. If X is a complex variety, $\mathbb{C}(X)$ denotes its function field.

We will identify vector bundles with their sheaves of sections.

1.1.2 For projective bundles we use Grothendieck's convention: if V is a vector bundle on a scheme X then we define $\mathbb{P}(V)$ as $\mathbf{Proj}\big(\mathrm{Sym}(V)\big)$, the relative homogeneous spectrum of the symetric algebra on V. (For more on homogeneous spectra, see Section 1.3.) This means that, for V a complex vector space of finite dimension, $\mathbb{P}(V)$ is the variety of hyperplanes in V, not the variety of 1-dimensional linear subspaces of V.

1.1.3 Unless indicated otherwise rings will be assumed to be commutative.

1.2 Divisors and line bundles

1.2.1 Definition. Let $\mathscr{L}$ be a line bundle on a scheme X. Then a global section $s \in \Gamma(X, \mathscr{L})$ is said to be *regular* is the homomorphism $\mathscr{O}_X \to \mathscr{L}$ given by $f \mapsto f \cdot s$ is injective.

1.2.2 We will freely use the notions and results explained in [13], Section II.6. If D is a Cartier divisor on a scheme X, we will write $\mathscr{O}_X(D)$ for the corresponding line bundle.

An *effective Cartier divisor* on a scheme X is a closed subscheme $D \subset X$ whose ideal sheaf $\mathscr{I}_D \subset \mathscr{O}_X$ is an invertible $\mathscr{O}_X$-module. If D is an effective Cartier divisor we define $\mathscr{O}_X(D)$ to be the dual $\mathscr{I}_D^{-1}$ of its ideal sheaf. The inclusion $\mathscr{I}_D \hookrightarrow \mathscr{O}_X$ defines a canonical global section 1_D of $\mathscr{O}_X(D)$, which is regular.

Effective Cartier divisors can be added; see [18], Tag 01WT. This makes the set of effective Cartier divisors a commutative monoid.

1.2.3 Proposition. *Let X be a scheme. Sending an effective Cartier divisor D to the pair $\big(\mathscr{O}_X(D), 1_D\big)$ gives a bijection between the set of Cartier divisors on X and the set of isomorphism classes of pairs $(\mathscr{L}, s)$ consisting of an invertible $\mathscr{O}_X$-module and a regular global section. The inverse map is obtained by sending $(\mathscr{L}, s)$ to the zero scheme of s.*

Proof. See [18], Tag 01X0. □

1.2.4 On a normal variety, effective Cartier divisors can also be regarded as (effective) Weil divisors. For this, suppose D is an effective Cartier divisor on the normal variety X. Let $Z \subset X$ be a prime divisor, i.e., an irreducible closed subscheme of codimension 1. Let ζ be the generic point of Z. By normality of X, the local ring $\mathscr{O}_{X,\zeta}$ is a discrete valuation ring. Let n_Z denote the length of $\mathscr{O}_{D,\zeta}$ as a module over $\mathscr{O}_{X,\zeta}$, or equivalently, the valuation of the image of 1_D in $\mathscr{O}_{X,\zeta}$. For a given D, the coefficient n_Z is non-zero for only finitely many prime divisors Z, and $\sum_Z n_Z \cdot Z$ is an effective Weil divisor. The map from effective Cartier divisors to effective Weil divisors thus obtained is injective.

1.3 Homogeneous spectra

One of the most basic tools in scheme theory is the Spec construction, which gives an one-to-one correspondence between commutative rings and affine schemes. For projective schemes, there is an analogue: taking the homogeneous spectrum $\mathrm{Proj}(R)$ of a graded ring R.

1.3.1 The *homogeneous spectrum* $\mathrm{Proj}(R)$ of a graded ring R is a scheme which as a set consists of the collection *homogeneous* prime ideals of R which do not contain the irrelevant ideal R_+, the ideal generated by all elements of positive degree. The closed subsets in $\mathrm{Proj}(R)$ are defined to be the sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ The stalk at a point $\mathfrak{p} \in \mathrm{Proj}(R)$ is just $\mathcal{O}_{\mathfrak{p}} = R_{(\mathfrak{p})}$, where $R_{(\mathfrak{p})}$ is the ring given by taking R and inverting all homogeneous elements not in $\mathfrak{p}$. For the precise definition of the sheaf structure on $\mathrm{Proj}(R)$, see [13, Chapter II, Section 2].

1.3.2 Remark. Note that this construction can be seen to be a generalisation of projective varieties, similarly to how affine spectra generalize affine varieties.

In later sections, we will also need a more general Proj construction, which works in a relative setting.

1.3.3 Let B be a scheme and $\mathscr{R}$ is a quasi-coherent sheaf of graded $\mathscr{O}_B$-algebras. If $U = \mathrm{Spec}(A)$ is an open affine subscheme of B, then $\mathscr{R}(U)$ is a graded A-algebra and we get a morphism $\mathrm{Proj}(\mathscr{R}(U)) \to U$. By gluing all schemes $\mathrm{Proj}(\mathscr{R}(U))$ for an open cover of affine subschemes U of B, we get a scheme $\mathbf{Proj}_B(\mathscr{R})$ together with a morphism $\pi\colon \mathbf{Proj}_B(\mathscr{R}) \to B$ such that for all open affine $U \subset B$ we have $\pi^{-1}(U) = \mathrm{Proj}(\mathscr{R}(U))$.

1.3.4 Remark. Note that the homogeneous spectrum of a graded ring R can just be seen as a special example of the above construction, where we just take $B = \mathrm{Spec}(\mathbb{Z})$ and view R as a graded $\mathbb{Z}$-algebra.

Projective space bundles are a special case of the above construction:

1.3.5 Definition. Let B be a scheme and $\mathscr{E}$ a locally free sheaf on B. We define the *projective space bundle associated to* $\mathscr{E}$ as

$$\mathbb{P}(\mathscr{E}) = \mathbf{Proj}_B(\mathscr{S}),$$

where $\mathscr{S} = S(\mathscr{E})$ is the symmetric algebra of $\mathscr{E}$. If $\mathscr{E}$ has rank $n+1$, then the fibre of the canonical morphism $\mathbb{P}(\mathscr{E}) \to B$ above any $b \in B$ is isomorphic to $\mathbb{P}^n$, so we will then call $\mathbb{P}(\mathscr{E})$ a $\mathbb{P}^n$*-bundle.*

Similarly to how a ring homomorphism $R \to S$ induces a morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ we also often get a rational map between the associated Proj:

1.3.6 Proposition. *Let B be a scheme and let $\varphi\colon \mathscr{R} \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_B$-algebras. For any affine open $U \subseteq B$ define the $G(\varphi_U) \subset \mathrm{Proj}(\mathscr{S}(U))$ to be the set of points not contained in the image of the irrelevant ideal:*

$$G(\varphi_U) = \{\mathfrak{p} \in \mathrm{Proj}(\mathscr{S}(U)) \mid \mathfrak{p} \not\supseteq \varphi(\mathscr{R}_+)\}.$$

Now define

$$G(\varphi) = \bigcup_{\substack{U \subseteq B,\\ U \text{ affine open}}} G(\varphi_U).$$

Then $G(\varphi)$ is an open subset of $\mathbf{Proj}_B(\mathscr{S})$, and φ defines an affine B-morphism

$$f : G(\varphi) \to \mathbf{Proj}_B(\mathscr{R})$$

which is given on an affine open $U \subseteq B$ by sending a homogeneous prime ideal $\mathfrak{p} \in \mathrm{Proj}(\mathscr{S}(U))$ to its inverse image $\varphi^{-1}\mathfrak{p}$. In particular, if $G(\varphi)$ is dense in B, the homomorphism φ induces a B-rational map

$$f\colon \mathbf{Proj}_B(\mathscr{S}) \dashrightarrow \mathbf{Proj}_B(\mathscr{R}).$$

Proof. This is shown in [11, Section 3.5.1]. □

Similar to the relative Proj construction, we can also consider the relative Spec construction.

1.3.7 Let B be a scheme and $\mathscr{R}$ is a quasi-coherent sheaf of $\mathscr{O}_B$-algebras. If $U = \mathrm{Spec}(A)$ is an open affine subscheme of B, then $\mathscr{R}(U)$ is an A-algebra and we get a morphism $\mathrm{Spec}(\mathscr{R}(U)) \to U$. By gluing all schemes $\mathrm{Spec}(\mathscr{R}(U))$ for an open cover of affine subschemes U of B, we get a scheme $\mathbf{Spec}_B(\mathscr{R})$ together with a morphism $\pi\colon \mathbf{Spec}_B(\mathscr{R}) \to B$ such that for all open affine $U \subset B$ we have $\pi^{-1}(U) = \mathrm{Spec}(\mathscr{R}(U))$.

For a polynomial algebra $\mathscr{R}[T]$ over an algebra $\mathscr{R}$, the homogeneous spectrum simply corresponds with the spectrum of $\mathscr{R}$:

1.3.8 Proposition. *Let B be a scheme and $\mathscr{R}$ a quasi-coherent $\mathscr{O}_B$-algebra, and set $\mathscr{R}[T] = \mathscr{R} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. Then there is an canonical isomorphism*

$$\mathbf{Proj}_B(\mathscr{R}[T]) \cong \mathbf{Spec}_B(\mathscr{R}).$$

In particular there is a canonical isomorphism

$$\mathbf{Proj}_B(\mathscr{O}_B[T]) \cong B.$$

Proof. See [11, Corollaire 3.1.7]. □

For some algebras, it is easier to work with elements of even degree. The following proposition shows that this subalgebra has the same homogeneous spectrum.

1.3.9 Proposition. *Let B be a scheme, let $\mathscr{R} = \bigoplus_{n=0}^{\infty} \mathscr{R}_n$ be a graded $\mathscr{O}_B$-algebra and let be d a positive integer. Define the subalgebra $\mathscr{R}^{(d)} = \bigoplus_{n=0}^{\infty} \mathscr{R}_{nd}$. Then the rational map*

$$\mathbf{Proj}_B(\mathscr{R}) \to \mathbf{Proj}_B(\mathscr{R}^{(d)})$$

induced by the inclusion $\mathscr{R}^{(d)} \to \mathscr{R}$ is a B-isomorphism.

Proof. See [11, Proposition 3.1.8 (i)]. □

Twisting the algebra using a line bundle also does not change the homogeneous spectrum, as the following proposition shows.

1.3.10 Proposition. *Let B be a scheme, let $\mathscr{R} = \bigoplus_{n=0}^{\infty} \mathscr{R}_n$ be a graded $\mathscr{O}_B$-algebra and let $\mathscr{L}$ be a line bundle on B. Define the graded $\mathscr{O}_B$-algebra $\mathscr{R}'$ by*

$$\mathscr{R}' = \bigoplus_{n=0}^{\infty} (\mathscr{R}_n \otimes_{\mathscr{O}_B} \mathscr{L}^{\otimes n}).$$

Then there is a B-isomorphism

$$\mathbf{Proj}_B(\mathscr{R}) \to \mathbf{Proj}_B(\mathscr{R}').$$

Proof. See [11, Proposition 3.1.8 (iii)]. □

The following construction will later be important to construct morphisms to homogeneous spectra.

1.3.11 Suppose $f\colon S \to B$ is a morphism of schemes, $\mathscr{L}$ a line bundle on S and $\mathscr{R}$ a graded quasi-coherent $\mathscr{O}_B$-algebra. Consider the quasi-coherent $\mathscr{O}_S$-algebra

$$\mathscr{S} \colon = \bigoplus_{n=0}^{\infty} \mathscr{L}^{\otimes n},$$

and suppose we have a morphism of graded $\mathscr{O}_S$-algebras

$$\psi\colon f^*(\mathscr{R}) \to \mathscr{S}.$$

Since f_* and f^* are adjoint functors, giving the morphism ψ is equivalent to giving a morphism of graded $\mathscr{O}_B$-algebras

$$\psi^\flat\colon \mathscr{R} \to f_*(\mathscr{S}).$$

By Proposition 1.3.8 and Proposition 1.3.10 we see that there is a canonical isomorphism

$$\mathbf{Proj}_S(\mathscr{S}) \cong S.$$

Therefore by Proposition 1.3.6 ψ induces a canonical B-morphism

$$\tau\colon G(\psi) \to \mathbf{Proj}_S(f^*\mathscr{R}),$$

where $G(\psi)$ is an open subscheme of S. Proposition (3.5.3) in [11] shows that there is an canonical isomorphism $\mathbf{Proj}_S(f^*\mathscr{R}) \to \mathbf{Proj}_B(\mathscr{R}) \times_B S$, and thus we have a canonical projection

$$\mathbf{Proj}_S(f^*\mathscr{R}) \to \mathbf{Proj}_B(\mathscr{R}).$$

Therefore, by taking the composition of τ and this projection, we find a canonical B-morphism

$$r_{\mathscr{L},\psi}\colon G(\psi) \to \mathbf{Proj}_B(\mathscr{R}).$$

The following proposition gives sufficient conditions to ensure $r_{\mathscr{L},\psi}$ is globally defined, meaning that $G(\psi) = S$ so we have a morphism

$$r_{\mathscr{L},\psi}\colon S \to \mathbf{Proj}_B(\mathscr{R}).$$

1.3.12 Proposition. *Under the assumptions of the above subsection, for $r_{\mathscr{L},\psi}$ to be globally defined, the following condition suffices: for every $x \in S$, there exists an integer $n > 0$ and a section $s \in \mathscr{R}_n \otimes k(b)$ such that for $t := \psi^\flat(s) \in f_*(\mathscr{L}^{\otimes n}) \otimes k(b)$ we have $t(x) \neq 0$ (where $b := f(x)$ and $k(b)$ is its residue field).*

Proof. For B affine and s a global section of $\mathscr{R}_n$, this is exactly [11, Corollaire (3.7.4)]. The general case follows from the observation that if $s \in \Gamma(V, \mathscr{R}_n)$ satisfies the conditions for an open $V \subset B$, then $s|_U$ also satisfies the condition for any affine open $U \subseteq V$. □

1.4 Double covers

We will only need to consider double covers of normal varieties, which somewhat simplifies the discussion.

1.4.1 Definition. Let Y be a normal complex variety. By a *double cover of* Y we then mean a finite surjective morphism $f\colon X \to Y$ where X is a normal complex variety, such that the extension of function fields $\mathbb{C}(Y) \subset \mathbb{C}(X)$ has degree 2.

1.4.2 Remark. Note that, since we require X to be a variety, the morphism $Y \amalg Y \to Y$ is not a double cover according to this definition.

1.4.3 Example. Let $X \subset \mathbb{A}^3$ be the affine variety given by the equation $z^2 = xy$. The projection $p\colon X \to Y = \mathbb{A}^2$ onto the first two coordinates realizes X as a double cover of the affine plane. Note that p is flat over $\mathbb{A}^2 \setminus \{O\}$ but not over the origin $O = (0, 0) \in \mathbb{A}^2$, and that X is singular in the unique point $(0, 0, 0)$ above O.

1.4.4 In what follows, we consider the quotient of a scheme by a finite group acting on it. This is a special case of a quotient of a scheme by a finite group scheme, which is treated in more detail in [10].

1.4.5 Definition. Let X be a scheme of finite type over $\mathbb{C}$, let G be a finite group, and let G act (on the left) on X by the homomorphism $\sigma : G \to \mathrm{Aut}(X)$. Then there exists a $\mathbb{C}$-scheme X/G and a morphism $q\colon X \to X/G$ over $\mathbb{C}$, unique up to unique isomorphism, with the following properties:

- the morphism q is G-invariant: for every $g \in G$ we have $q \circ \sigma_g = q$,

- among all G-invariant morphisms, q is universal: for every scheme Y over $\mathbb{C}$ and every G-invariant morphism $f\colon X \to Y$, there exists an unique morphism $g\colon X/G \to Y$ such that $f = g \circ q$.

We call X/G the *quotient of X by G* and q the *quotient map.*

The following are basic properties of such quotients:

1.4.6 Proposition. *Let X, G, σ and q be as in the above definition. Then the following properties hold:*

- *the scheme X/G is of finite type over $\mathbb{C}$, and the morphism q is finite and surjective,*
- *the topology on X/G is the quotient topology given by q,*
- *the fibres of q are the G-orbits in X,*
- *there is a natural isomorphism $\mathscr{O}_{X/G} \cong (q_*\mathscr{O}_X)^G$.*

Proof. These statements are special cases of [10, Proposition 4.8]. □

From the properties of quotient schemes, we directly obtain the following proposition.

1.4.7 Proposition. *Let X be a normal variety, and ι an involution on X. Then the morphism*

$$X \to X/\langle\iota\rangle$$

is a double cover.

Thus an involution on a surface induces a double cover. In the following lemma we see the converse also holds.

1.4.8 Lemma. *Let $f\colon X \to Y$ be a double cover of a normal variety Y. Then X is the normalisation of Y in the function field $\mathbb{C}(X)$. There is a unique involution $\iota\colon X \to X$ of X over Y such that the induced automorphism of $\mathbb{C}(X)$ is the unique non-trivial automorphism of $\mathbb{C}(X)$ over $\mathbb{C}(Y)$, and the canonical morphism $X/\langle\iota\rangle \to Y$ is an isomorphism.*

Proof. The first claim directly follows from the fact that X is normal. The normalisation of Y in $\mathbb{C}(X)$ is clearly sent to itself by the non-trivial involution of $\mathbb{C}(X)$ over $\mathbb{C}(Y)$, so we get an involution ι on X. Therefore we find that the morphism factors through the morphism $X/\langle\iota\rangle \to Y$. This morphism is finite and an isomorphism on function fields, so since Y is normal it is an isomorphism. □

1.4.9 Let $f\colon X\to Y$ be a double cover, as in Definition 1.4.1. The involution ι of X over Y induces an automorphism of order 2 of $f_*(\mathscr{O}_X)$ as an $\mathscr{O}_Y$-module, which we again call ι. Since 2 is invertible in our base field and $X/\langle\iota\rangle \xrightarrow{\sim} Y$ this induces a decomposition

$$f_*(\mathscr{O}_X) = \mathscr{O}_Y \oplus f_*(\mathscr{O}_X)^-\,, \tag{1.4.9.1}$$

where $f_*(\mathscr{O}_X)^- \subset f_*(\mathscr{O}_X)$ is the subsheaf of anti-invariant sections.

1.4.10 Suppose $f\colon X\to Y$ is a *flat* double cover. In this case, $f_*(\mathscr{O}_X)$ is locally free of rank 2 as an $\mathscr{O}_Y$-module, and it follows that $f_*(\mathscr{O}_X)^-$ is a line bundle. Define $\mathscr{L}$ to be the inverse of this line bundle, so that (1.4.9.1) now reads: $f_*(\mathscr{O}_X) = \mathscr{O}_Y \oplus \mathscr{L}^{-1}$. Because $f_*(\mathscr{O}_X)$ has the structure of an $\mathscr{O}_Y$-algebra, and the product of two anti-invariant sections is invariant, we have a morphism

$$\mathscr{L}^{-1}\otimes\mathscr{L}^{-1}\to\mathscr{O}_Y\,.$$

Write s for the corresponding global section of $\mathscr{L}^2$. The algebra structure of $f_*(\mathscr{O}_X)$ is then given by the rule

$$(a_1+b_1)\cdot(a_2+b_2) = (a_1a_2 + s\cdot b_1b_2) + (a_1b_2 + a_2b_1)\,,$$

for a_1, a_2 local sections of $\mathscr{O}_Y$ and b_1, b_2 local sections of $f_*(\mathscr{O}_X)^- = \mathscr{L}^{-1}$. As $X = \mathbf{Spec}\big(f_*(\mathscr{O}_X)\big)$ (because f is finite, and hence affine), we see that the double cover is completely determined by the pair $(\mathscr{L}, s)$. It is a classical result that, via this construction, isomorphism classes of flat double covers of Y (assumed to be normal) are in bijective correspondence with the isomorphism classes of pairs $(\mathscr{L}, s)$ consisting of a line bundle $\mathscr{L}$ and a non-zero global section of $\mathscr{L}^2$.

The pull-back $f^*(s)$ of s to a global section of $f^*(\mathscr{L})$ is a square: if

$$t = (0,1)\in\Gamma(Y,\mathscr{L})\oplus\Gamma(Y,\mathscr{O}_Y) = \Gamma(Y, f_*f^*\mathscr{L}) = \Gamma(X, f^*\mathscr{L})$$

then $t^2 = f^*(s)$.

Because X is a variety (and therefore an integral scheme) we have $s\neq 0$ and hence also $t\neq 0$. Because Y is an integral scheme as well, it follows that s is a regular section of $\mathscr{L}^2$ and t is a regular section of $f^*\mathscr{L}$ (see Definition 1.2.1). This justifies the next definition.

1.4.11 Definition. Let $f\colon X\to Y$ be a flat double cover. Then we define the *branch divisor* of f (or of X over Y) to be the effective Cartier divisor Br_f on Y obtained as the zero scheme of the section $s\in\Gamma(Y,\mathscr{L}^2)$. We define the *ramification divisor* of f to be the effective Cartier divisor R_f on X obtained as the zero scheme of the section $t\in\Gamma(X, f^*\mathscr{L})$.

In other words, Br_f is the effective Cartier divisor on Y that, under the correspondence of Proposition 1.2.3, corresponds to the pair $(\mathscr{L}^2, s)$, and R_f is the effective Cartier divisor on X that corresponds to $(f^*\mathscr{L}, t)$. By construction, $f^*(\mathrm{Br}_f) = 2 \cdot \mathrm{R}_f$. The support of the ramification divisor is the locus of points where f is ramified, i.e., the support of the sheaf $\Omega_{X/Y}$ of relative differentials. We want to extend these notions to double covers which are not necessarily flat. For this we first need the following proposition.

1.4.12 Proposition. *Let $f\colon X \to Y$ be a double cover of normal varieties. Then there is a largest open $Y^\circ \subset Y$ such that f is flat over Y°. Furthermore, the complement $Y - Y^\circ$ has codimension at least 2 in Y. In particular, if Y is a surface, f is flat over the complement of a finite set of points.*

Proof. Consider a point $y \in Y$ in codimension 1. Since the local ring $\mathscr{O}_{Y,y}$ is a discrete valuation ring, a finitely generated module M over $\mathscr{O}_{Y,y}$ is flat if and only if it is torsion-free. Thus if t is an uniformizer at y, then M is flat if and only if multiplication by t is injective on M.

If $x \in X$ is a point such that $f(x) = y$, then $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat, because $\mathscr{O}_{X,x}$ is a subring of the function field $\mathbb{C}(X)$. Thus we see f is flat in codimension 1, as desired. □

Now assume Y is a surface. If we define $X^\circ = f^{-1}(Y^\circ)$, the morphism $f\colon X^\circ \to Y^\circ$ is a flat double cover, and we have a branch divisor Br_f° on Y° and a ramification divisor R_f° on X°. Let Br_f (resp. R_f) be the Weil divisors on Y (resp. on X) obtained by taking the closure of Br_f° (resp. R_f°).

1.4.13 Proposition. *Let $f\colon X \to Y$ be a double cover of normal surfaces. Let* $\mathrm{Ram}(f) \subset X$ *be the ramification locus of f, i.e., the closed set of points $x \in X$ where f is ramified (in the sense that $\Omega_{X/Y,x} \neq 0$). Then* $\mathrm{Ram}(f)$ *equals the support of* R_f *together with a (possibly empty) finite set of points contained in $f^{-1}(Y^{\mathrm{sing}})$.*

Proof. It is clear that $\mathrm{Ram}(f) \cap X^\circ$ equals the support of R_f°, and because $\mathrm{Ram}(f)$ is closed, it follows that $\mathrm{Ram}(f)$ equals the union of the support of R_f together with a (necessarily finite) set of points contained in $X \setminus X^\circ$. If $x \in \mathrm{Ram}(f)$ is not contained in $X^\circ \cup |\mathrm{R}_f|$ then its image $y \in Y$ must be a singular point by purity of the branch locus, see [18], Tag 0BMB. □

Furthermore the singularities of a double covering of a smooth surface only occur on singular points of the ramification locus, as the following lemma shows.

1.4.14 Lemma. *Let $f : X \to Y$ be a double cover of normal surfaces, with Y smooth. If the ramification divisor* R_f *is a smooth curve, then X is smooth.*

Proof. This follows from [5, Chapter III, Theorem 5.2]. □

Chapter 2

Basic notions from the theory of algebraic surfaces

2.1 Basic definitions

One of the most important tools in working with algebraic surfaces is the intersection pairing on a smooth surface. This is a pairing on divisors which generalizes the notion of intersection multiplicity on the projective plane, and can be characterised by simple conditions as given in the following proposition:

2.1.1 Proposition. *Let S be a smooth projective surface and let* Div S *denote the group of divisors on S. There is an unique pairing* Div $S \times$ Div $S \to \mathbb{Z}$*, which is called the* intersection pairing *and is denoted by $C \cdot D$ for divisors C, D, such that*

(1) *if C and D are curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$.*
(2) *it is symmetric: $C \cdot D = D \cdot C$*
(3) *it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$*
(4) *it depends only on the linear equivalence class: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.*

Proof. See [13, Chapter V, Theorem 1.1]. □

For any divisor D we denote its self-intersection by $D^2 := D \cdot D$.

2.1.2 Definition. A curve C on a smooth projective surface S is called a (-1)-*curve* if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

We can wonder whether any (-1)-curve can be seen as an exceptional curve for a blowup of some surface. A celebrated theorem of Castelnuovo shows that we can do this:

2.1.3 Proposition (Castelnuovo)**.** *Let C be a (-1)-curve on a smooth projective surface S. Then there exists a morphism $f\colon S \to S_0$ to a smooth projective surface S_0, and a point $P \in S_0$ such that S is isomorphic via f to the blowup of S_0 at P, and C is the exceptional curve.*

Proof. See [13, Chapter V, Theorem 5.7]. □

So any (-1)-curve on a surface can be contracted. This leads to the notion of a minimal model.

2.1.4 Definition. Let S be a smooth surface. If S has no (-1)-curves we say that S is *minimal*. Furthermore if Y is a minimal surface birational to S, then we say Y is a *minimal model* for S.

It turns out that for any smooth surface a minimal model exists:

2.1.5 Proposition. *Let S be a smooth projective surface. There exists a minimal model Y of X, together with a birational morphism*

$$\pi\colon S \to Y.$$

Proof. See [13, Chapter V, Theorem 5.8]. □

In fact, for surfaces which are not rational or ruled, there is even an *unique* minimal model.

Blowups are useful for understanding birational maps between surfaces, since by the following proposition all birational morphisms can be described using blowups:

2.1.6 Proposition. *Let $f\colon S \to Y$ be a birational morphism between smooth projective surfaces. Then f can be decomposed as a sequence of morphisms*

$$S \cong S_n \to S_{n-1} \to \cdots \to S_0 \cong Y,$$

where $S_i \to S_{i-1}$ is the morphism obtained by blowing up a single point of S_{i-1}.

Proof. This is a special case of [13, Chapter V, Theorem 5.5]. □

If we combine this proposition with Castelnuovo's result we immediately see that the only irreducible fibres of birational morphisms are (-1)-curves:

2.1.7 Corollary. *Let $f\colon S \to Y$ be a birational morphism between smooth projective surfaces and $y \in Y$. If the fibre $C := f^{-1}(y)$ is irreducible, then C is a (-1)-curve.*

2.2 Rational singularities

In general, surfaces can have very complicated singularities, which makes their study much harder. However, many varieties only have mild singularities called *rational singularities*, which we will define in this section.

Suppose we have an isolated singularity x on a surface X. By work of Hironaka, we can resolve the singularity: there exists a nonsingular surface S and a birational proper morphism $\pi\colon S \to X$.

2.2.1 Definition. We call a nonzero effective divisor $Z = \sum_{i=0}^{n} r_i E_i \subset S$ a *fundamental cycle for x* if

$$Z \cdot E_i \leq 0 \quad \text{for all} \quad i,$$

and for any divisor $Z' = \sum_{i=0}^{n} r_i' E_i \subset S$ which also satisfies

$$Z' \cdot E_i \leq 0 \quad \text{for all} \quad i,$$

we have $r_i' \geq r_i$ for all i.

2.2.2 Lemma. *Let X be a surface, $x \in X$ an isolated singularity and $\pi\colon S \to X$ be a resolution of singularities. There exists a unique fundamental cycle Z for x. Furthermore we have $p_a(Z) \geq 0$, where $p_a(Z)$ is the arithmetic genus of Z.*

Proof. This is shown in the work of Artin [2, Theorem 3]. □

Now we can finally define what it means for a singularity to be rational.

2.2.3 Definition. Let X be a surface, $x \in X$ an isolated singularity, $\pi\colon S \to X$ be a resolution of singularities and Z the fundamental cycle for x. We say that x is a *rational singularity* if the fundamental cycle satisfies $p_a(Z) = 0$. If furthermore the multiplicity of x is equal to 2, then we call x a *rational double point.*

A priori, it may seem like the rationality of x depends on the chosen resolution of singularities, but Artin [2] shows this is not the case.

Analogously to the notion of a (-1)-curve, we also introduce the notion of a (-2)-curve:

2.2.4 Definition. Let S be a surface and C a curve on S. We call C a (-2)-curve if $C^2 = -2$ and $C \cong \mathbb{P}^1$.

It is known that all the components of a minimal resolution of a rational singularity are (-2)-curves.

2.2.5 Proposition. *Let X be a surface, $x \in X$ a rational singularity and let $\pi\colon S \to X$ be a resolution of singularities which is minimal: the fibres of π do not contain (-1)-curves. Then the irreducible components $E_1, \dots E_n$ of $\pi^{-1}(x)$ are all (-2)-curves. Furthermore, for $i \neq j$ the intersection of E_i and E_j is transverse, and no three E_i meet in a point.*

Proof. This follows from the equivalence of Characterisation A2 and Characterisation A3 in [9]. □

The next proposition shows that for rational singularities, the self intersection of a fundamental cycle is determined by the multiplicity of the singularity.

2.2.6 Proposition. *Let X be a surface, $x \in X$ an isolated rational singularity, and Z a fundamental cycle for x for some resolution of singularities. Then the multiplicity of the singularity is given by $-(Z^2)$. In particular, if x is a rational double point, then we have $Z^2 = -2$.*

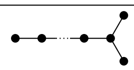
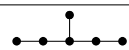
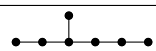
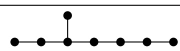
Proof. See [2, Corollary 6]. □

In the remainder of the section, we will discuss the classification of rational double points, as well as their double covers.

2.2.7 It is well known [7] that any rational double point on a surface is analytically isomorphic to the origin of the quotient variety $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Up to conjugation, a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is one of the following:

- The cyclic group $\mathscr{C}_n$ of order n,
- the binary dihedral group $\mathscr{D}_n$ of order $4n$,
- the binary tetrahedal group $\mathscr{T}$ of order 24,
- the binary octahedral group $\mathscr{O}$ of order 48,
- the binary icosahedral group $\mathscr{I}$ of order 120.

Therefore we see there are 2 families of (analytical) isomorphism classes of rational double points, along with 3 exceptional cases. In the following table the names of these classes of rational double points are given, as well as a simple equation whose zero locus is an affine surface in $\mathbb{A}^3$ having such a rational double point at the origin.

Name	Equation	Group	Dynkin diagram
A_n, $n \geq 1$	$x^2 + y^2 + z^{n+1}$	$\mathscr{C}_{n+1}$	
D_n, $n \geq 4$	$x^2 + y^2 z + z^{n-1}$	$\mathscr{D}_{n-2}$	
E_6	$x^2 + y^3 + z^4$	$\mathscr{T}$	
E_7	$x^2 + y^3 + yz^3$	$\mathscr{O}$	
E_8	$x^2 + y^3 + z^5$	$\mathscr{I}$	

In this table, the Dynkin diagrams of A_n, D_n and E_n have exactly n edges. The vertices in the Dynkin diagram represent the (-2)-curves above the rational double point in a minimal resolution of singularities, while two vertices share an edge if they intersect each other.

Earlier we discussed double covers of varieties, which we will now apply to study covers of rational double points. First we define the notion of an unramified covering of a rational double point.

2.2.8 Definition. Let $f\colon X \to Y$ be a double cover of a variety Y, and let $y \in Y$ be a rational double point. We say that f is *unramified* near y if y does not lie in the divisor R_f. Equivalently, there is an analytic open $U \subset Y$ containing y such that $f|_{f^{-1}(U-\{y\})}$ is unramified.

The points above a rational double point are also rational double points:

2.2.9 Proposition. *Let $f\colon X \to Y$ be double cover which is unramified near a rational double point $y \in Y$. If $x \in X$ satisfies $f(x) = y$ then x is smooth or also a rational double point.*

Proof. If $f^{-1}(y) = \{x\}$ then this is [3, Proposition 1.4]. If $f^{-1}(y)$ consists of two points instead, then there is an analytic open $U \subset Y$ containing y such that $f|_{f^{-1}(U)}$ is unramified, so we see that any point in $f^{-1}(y)$ is a rational singularity of the same type as y. □

If we consider a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ and a subgroup $G' \subset G$ of index 2 then we get a double cover

$$\mathbb{C}^2/G' \to \mathbb{C}^2/G,$$

which is clearly an unramified double cover of the rational double point $0 \in \mathbb{C}^2/G$. Conversely, if we have a double cover

$$\mathbb{C}^2/G' \to \mathbb{C}^2/G,$$

where G and G' are finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ then we can view G' as an index 2 subgroup of G. Since any rational double point y is analytically isomorphic to the origin of $\mathbb{C}^2/G$ for G a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, we see that this gives all possible types of rational double points above y. Using this idea, Artin [3] compiled a list of all such double covers:

2.2.10 Proposition. *Let $f\colon X \to Y$ be an double cover of a rational double point $y \in Y$ which is unramified near y with $f^{-1}(y) = \{x\}$. Then the cover is of one of these forms, for $k \geq 1$*

- $A_{k-1} \to A_{2k-1}$ *induced by the inclusion* $\mathscr{C}_k \subset \mathscr{C}_{2k}$,

- *$A_{2k-1} \to D_{k+2}$ induced by the inclusion $\mathscr{C}_{2k} \subset \mathscr{D}_k$,*
- *$D_{k+2} \to D_{2k+2}$ induced by the inclusion $\mathscr{D}_k \subset \mathscr{D}_{2k}$,*
- *$E_6 \to E_7$ induced by the inclusion $\mathscr{T} \subset \mathscr{O}$,*

where we interpret A_0 as a smooth point and D_3 as A_3.

2.3 The canonical model

In this section we study the canonical model of a surface of general type, which can be seen a two dimensional analogue of the notion of a canonical image of a curve. We start by briefly recalling the notion of the canonical ring of a surface and its type.

2.3.1 Definition. Let S be a smooth projective variety, and denote its canonical bundle by K_S. The *canonical algebra* of the surface S is defined as the graded ring

$$R = \bigoplus_{m=0}^{\infty} H^0(S, mK_S).$$

We call $\kappa(S) := -1 + \operatorname{trdeg}_{\mathbb{C}}(R)$ the Kodaira dimension of S.

For a surface, the canonical algebra is finitely generated:

2.3.2 Proposition. *Let S be a smooth projective surface. The canonical algebra R of the surface S is a finitely generated graded ring.*

Proof. This is the last part of [4, Theorem 9.1]. □

It is a well-known fact that for any smooth projective variety S of dimension n, the Kodaira dimension $\kappa(S)$ lies between -1 and n.

2.3.3 Definition. Let S be a smooth projective variety. If $\kappa(S) = n$ we say that S is of *general type*, while if $\kappa(S) \leq n-1$ we say that S is of *special type*.

A curve is of general type if and only if its genus is at least 2, while for surfaces Kodaira [14] proved similar criterion for determining whether a surface is of general type:

2.3.4 Proposition. *Let S be a minimal smooth projective surface, and denote its canonical bundle by K_S. Then S is of general type if and only if $K_S^2 \geq 1$ and $P_2 \geq 2$, where*

$$P_2 = \dim_{\mathbb{C}} H^0(S, 2K_S).$$

We will now define the canonical model of a surface of general type.

2.3.5 Definition. Let S be a smooth projective variety of general type of dimension n. The canonical model of S is defined as the n-dimensional $\mathbb{C}$-scheme

$$X = \mathrm{Proj}(R),$$

where R is the canonical algebra of S.

If a tensor power of the canonical sheaf is generated by global sections, then we can construct a morphism to the canonical model. For a curve C of genus at least 2, we know by [13, Chapter IV, Corollary 3.2] that $\omega_C^{\otimes n}$ is generated by global sections for $n \geq 3$. For surfaces of general type, Bombieri proved a similar result:

2.3.6 Lemma. *Let S be a smooth projective surface of general type. Then for $n \geq 4$ the line bundle $\omega_S^{\otimes n}$ is generated by global sections.*

Proof. See [6, Theorem 2]. □

Using the previous lemma we see that there is a natural morphism to the canonical model:

2.3.7 Proposition. *Let S be a smooth projective curve or surface of general type, with canonical model X. Then there exists a natural morphism $\pi\colon S \to X$.*

Proof. Let $f : S \to \mathrm{Spec}(\mathbb{C})$ be the structure morphism and set $\mathscr{R}_n = f_*\omega_S^{\otimes n} = H^0(S, nK_S)$ as well as $\mathscr{L} = \omega_S$ and $\mathscr{S} = \oplus_{n=0}^{\infty}\mathscr{L}^{\otimes n}$. We have a natural injection $\psi\colon f^*f_*\mathscr{S} \to \mathscr{S}$ given by the inclusion of global sections. Since this is the map induced by adjunction, we have $\psi^\flat = \mathrm{id}$. Now Proposition 1.3.12 implies that we have a morphism

$$r_{\mathscr{L},\psi} : S \to X$$

since $\omega_S^{\otimes 4}$ is generated by global sections by 2.3.6. Therefore we take $\pi = r_{\mathscr{L},\psi}$. □

The morphism π to the canonical model is a birational map, as the following theorem shows:

2.3.8 Theorem. *Let S be a minimal smooth projective surface of general type and let $\pi\colon S \to X$ be the morphism to its canonical model. The canonical model X has at most rational double points as singularities, and $\pi\colon S \to X$ is a minimal resolution of singularities of X, which contracts all (-2)-curves in S. In particular we see π is a birational morphism.*

Proof. Let $C \subset S$ be a (-2)-curve. The adjunction formula shows that we must have $\deg_C(K_S|_C) = C \cdot K_S = 0$, so we see the restriction of ω_S to C is a degree 0 line bundle, so the restriction is the trivial bundle. Thus for any n, any section in $\omega_S^{\otimes n}(S)$ is constant on C. Therefore C must be mapped to a point by π. The other statements follow directly from [4, Theorem 9.1]. □

2.4 Fibrations

In order to better understand a surface, it helps to have a non-constant map from the surface to a curve. Such a map will then exhibit the surface as a 1-dimensional family of curves, and using this a problem for a surface can sometimes be reduced to a problem for a curve. A special example of such a map is given by a fibration.

2.4.1 Definition. Let S be a smooth projective surface and B a smooth projective curve. A *fibration* from S to B is a morphism

$$f\colon S \to B,$$

such that f is not constant and its fibres are connected. We denote the fibre of f above $b \in B$ as $F_b := f^{-1}(b)$.

Any fibre of such a fibration has the same genus:

2.4.2 Proposition. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. Then the fibres of f are one dimensional, and the arithmetic genus $g = p_a(F_b)$ does not depend on the choice of $b \in B$.*

Proof. Since B is a smooth curve and f is not a constant morphism, it follows from [13, Chapter II, Proposition 9.7] that the morphism f is flat. Using the flatness of f, the desired statements follow from [13, Chapter III, Proposition 9.5] and [13, Chapter III, Corollary 9.10]. □

The former proposition leads to the following definition:

2.4.3 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. For any $b \in B$ we call the constant $p_a(F_b)$ the *genus of the fibration.*

Most fibres of a fibration are smooth, as the following proposition shows.

2.4.4 Proposition. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. For all but finitely many $b \in B$, the fibre F_b is smooth.*

Proof. Since S is smooth and f is dominant, the statement directly follows from [13, Chapter III, Corollary 10.7]. □

We will now illustrate the concept by giving some examples.

2.4.5 Example. Any geometrically ruled surface is an example of a fibration of genus 0. However not every fibation of genus 0 is a ruled surface, since not all fibres need to be smooth.

2.4.6 Example. Fibrations of genus 1 are also known as *elliptic surfaces*, which have a well studied theory.

2.4.7 Example. If we consider the projective surface S in $\mathbb{P}^2 \times_{\mathbb{C}} \mathbb{P}^1$ defined on an affine chart by $y^2 = f(x;t)$ with $f(x;t) \in \mathbb{C}(t)[x]$, then the projection $f\colon S \to \mathbb{P}^1$ defined by $(x, y; t) \mapsto t$ has connected fibres. By repeatedly blowing up the singular points of S, we arrive at a smooth projective surface $\widetilde{S}$ with a fibration

$$\widetilde{f}\colon \widetilde{S} \to \mathbb{P}^1.$$

The genus of this fibration is g, where $\deg_x(f(x;t)) = 2g + 1$ or $\deg_x(f(x;t)) = 2g + 2$. If $g \geq 2$ then this gives an example of a *hyperelliptic fibration*, meaning that the general fibre is hyperelliptic.

2.4.8 Example. If we take $f(x;t) = x^2 + t$ in the previous example, we see that fibres do not need to be irreducible, since the fibre at $t = 0$ is locally on an affine chart the zero locus of $y^2 - x^2$, which is clearly reducible.

2.4.9 Example. In a similar vein, by taking $f(x;t) = tx$ we also see that fibres need not be reduced, since the fibre at $t = 0$ is locally on an affine chart the zero locus of y^2, which is clearly not reduced.

For simplicity, it is useful to associate to a fibration $f\colon S \to B$ a sort of minimal model of the fibration. A first thought could be to take a minimal model S' for the surface S, and consider the induced rational map $f'\colon S' \to B$. However f' is not a morphism if a (-1)-curve C contracted in $S \to S'$ does not map constantly to B under f. Therefore we arrive at the following definition:

2.4.10 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. Then f is called a *relatively minimal fibration* if none of the fibres contain (-1)-curves.

One of the most important invariants of a fibration is its relative canonical bundle:

2.4.11 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration, and denote the canonical (or dualizing) sheaves on them by ω_S and ω_B. The *relative dualizing sheaf* $\omega_{S/B}$ is defined as the difference of the two dualizing sheaves:

$$\omega_{S/B} := \omega_S \otimes f^*\omega_B^{-1}.$$

For the associated divisor class we will write $K_{S/B}$.

2.4.12 In what follows, we need the notion of a *dualizing sheaf*. The dualizing sheaf of a projective Cohen-Macaulay scheme X is a coherent sheaf ω_X on X, which coincides with the canonical sheaf if X is smooth. If ω_X is a line bundle, we will also write K_X for its associated divisor class. For the precise definition of the dualizing sheaf see [13, Chapter III, §7]. For the dualizing sheaf on a fibre we have a form of Serre duality:

2.4.13 Proposition. *Let S be a smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration and $b \in B$. Then ω_{F_b} is a line bundle satisfying*

$$H^i(F_b, \omega_{F_b}) \cong H^{1-i}(F_b, \mathcal{O}_{F_b})^\vee, \quad \text{for all} \quad i,$$

where $\vee$ refers to taking the dual.

Proof. Since the inclusion $F_b \subset S$ is locally given by a single equation, it follows from [18, tag0BVA] that the fibre F_b is Gorenstein so by [18, tag0BFQ] the dualizing sheaf ω_{F_b} is a line bundle. Since F_b is Gorenstein, it is also Cohen-Macaulay, so [13, Chapter III, Corollary 7.7] applies, giving the desired isomorphism. □

The dualizing sheaf of F_b is simply the restriction of the canonical sheaf of S:

2.4.14 Proposition. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. The the restriction of the relative dualizing sheaf to a fibre is given by the dualizing sheaf of the fibre:*

$$\omega_{S/B}|_{F_b} \cong \omega_{F_b},$$

where the restriction is defined by $\omega_{S/B}|_{F_b} := i^*\omega_{S/B}$, where i is the inclusion of the fibre into S.

Proof. First we note that $f^*\omega_B|_{F_b} = (f \circ i)^*\omega_B = \mathscr{O}_{F_b}$ since $f \circ i$ the constant map to b. Therefore we see that $\omega_{S/B}|_{F_b} = \omega_S|_{F_b}$.

We have by [1, Chapter I, Proposition 2.4] the isomorphism

$$\omega_{F_b} \cong \mathscr{O}_{F_b} \otimes_{\mathscr{O}_S} (\mathscr{O}_S(F_b) \otimes_{\mathscr{O}_S} \omega_S)$$

and since $\mathscr{O}_{F_b} \otimes_{\mathscr{O}_S} \mathscr{O}_S(F_b) \cong \mathscr{O}_{F_b}$, we therefore have

$$\omega_{F_b} \cong \mathscr{O}_{F_b} \otimes_{\mathscr{O}_S} \omega_S,$$

which was the desired isomorphism. □

In the next section the pushforward of powers of the dualizing sheaf will play a big role, so we introduce the following notation.

2.4.15 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration. For $n \geq 0$ we define $\mathscr{R}_n$ to be the $\mathscr{O}_B$-module defined as follows

$$\mathscr{R}_n := f_*(\omega_{S/B}^{\otimes n}).$$

These sheaves are locally free of known rank:

2.4.16 Lemma. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration of genus $g \geq 2$. For any $n \geq 0$, the sheaf $\mathscr{R}_n$ is locally free. The sheaf $\mathscr{R}_1$ has rank g, while for $n > 1$ the sheaf $\mathscr{R}_n$ has rank $(2n-1)(g-1)$. Furthermore, for any $b \in B$ the natural map*

$$\mathscr{R}_n \otimes k(b) \to H^0(F_b, \omega_{F_b}^{\otimes n})$$

is an isomorphism, where we write $k(b)$ for the residue field of b.

Proof. Since f is flat and $\omega_{S/B}^{\otimes n}$ is locally free, we see $\mathscr{R}_n$ is flat over $\mathscr{O}_B$. Note that morphism f is projective and thus proper. Therefore, using [12, Théorème (7.9.4)] with $X = S$, $Y = B$, $M = \mathscr{O}_B$ and $P = \omega_{S/B}^{\otimes n}$ (seen as complex with a single term at degree 0) we see that the Euler characteristic $\chi(\omega_{F_b}^{\otimes n})$ does not depend on the choice of $b \in B$. Therefore since $h^0(F_b, \mathscr{O}_{F_b}) = 1$ for all $b \in B$ (since the fibres are connected), the isomorphisms from Proposition 2.4.13 imply

$$h^0(F_b, \omega_{F_b}) = g, \quad h^1(F_b, \omega_{F_b}) = 1, \quad \text{for all } b \in B.$$

Combining this with Grauert's theorem [13][Chapter III, Corollary 12.9] we see that $\mathscr{R}_1$ is locally free of rank g and we get the desired isomorphism.

Now assume $n > 1$. We have $h^0(F_b, \omega_{F_b}) = g \geq 2$, and thus $h^0(F_b, \omega_{F_b}^{\otimes m}) \geq 2$ for all $m > 0$. We first use this to prove $H^0(F_b, \omega_{F_b}^{\otimes -m}) = 0$. If $0 \neq \alpha \in H^0(F_b, \omega_{F_b}^{\otimes -m})$ and $\beta, \gamma \in H^0(F_b, \omega_{F_b}^{\otimes m})$ are two linearly independent sections, then $\alpha \cdot \beta$ and $\alpha \cdot \gamma$ are two linearly independent global sections of $H^0(F_b, \mathcal{O}_{F_b})$. But $H^0(F_b, \mathcal{O}_{F_b})$ is one dimensional so we have a contradiction. Thus $H^0(F_b, \omega_{F_b}^{\otimes -m}) = 0$. Using Serre duality we therefore see $H^1(F_b, \omega_{F_b}^{\otimes n}) = 0$ for all $b \in B$. Since the Euler characteristic $\chi(\omega_{F_b}^{\otimes n})$ does not depend on $b \in B$, we therefore see $h^0(F_b, \omega_{F_b}^{\otimes n})$ is also independent of the choice of $b \in B$. We can therefore compute this constant on a smooth fibre using Riemann-Roch, from which we conclude

$$h^0(F_b, \omega_{F_b}^{\otimes n}) = (2n-1)(g-1), \quad h^1(F_b, \omega_{F_b}^{\otimes n}) = 0, \quad \text{for all } b \in B.$$

As in the case with $n = 1$, we can combine this with Grauert's theorem to see $\mathscr{R}_n$ is locally free of rank $(2n-1)(g-1)$ and we get the desired isomorphism. □

2.5 The relative canonical model

In this section we will introduce the relative canonical model of a fibration. This construction greatly resembles the canonical model, but it has several advantages. For studying fibrations $f\colon S \to B$, the main issue of the canonical model is that f in general only induces a rational map $f'\colon X \dashrightarrow B$ from the canonical model instead of a morphism. Another drawback of the canonical model is that it only exists for surfaces of general type. The relative canonical model will have neither of these problems: it both respects the fibration structure and it is defined for all fibrations, even for surfaces of special type.

2.5.1 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a fibration of genus $g \geq 2$. We define the *relative canonical algebra* to be the graded $\mathscr{O}_B$-algebra

$$\mathscr{R} := \bigoplus_{n=0}^{\infty} \mathscr{R}_n$$

and define the *relative canonical model* to be its homogeneous spectrum:

$$\mathfrak{X} := \mathbf{Proj}_B(\mathscr{R}).$$

For the canonical model, we used that $\omega_S^{\otimes n}$ is generated by global sections to construct the morphism $\pi : S \to X$. For the relative canonical model, we use a similar result on the second tensor power of the dualizing sheaves $\omega_{F_b}^{\otimes 2}$ to construct a morphism $\sigma : S \to \mathfrak{X}$.

2.5.2 Lemma. *Let S be a smooth projective surface, B a smooth projective curve, $b \in B$ and $f\colon S \to B$ a fibration of genus $g \geq 2$. Then the sheaf $\omega_{F_b}^{\otimes 2}$ is generated by global sections.*

Proof. This is proven by Mendes Lopes in [15, Chapter I, Proposition 8.13]. □

Using this result we can prove the following:

2.5.3 Lemma. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$ and $\mathfrak{X}$ its relative canonical model. There exists natural morphism*

$$\sigma\colon S \to \mathfrak{X}.$$

Proof. Since f_* and f^* are adjoint functors, note that the identity map $\psi^\flat = \mathrm{id}\colon \mathscr{R} \to \mathscr{R}$ induces a morphism of $\mathscr{O}_S$-algebras

$$\psi\colon f^*\mathscr{R} \to \mathscr{S} = \bigoplus_{n=0}^{\infty} \omega_{S/B}^{\otimes n}.$$

Now Proposition 1.3.12 with $\mathscr{L} = \omega_{S/B}$ implies that we have a morphism

$$r_{\mathscr{L},\psi} : S \to \mathfrak{X}$$

so long as for every $x \in S$ there is a section $s \in \mathscr{R}_2 \otimes k(b)$, where $b = f(x)$, such that $s(x) \neq 0$. By Lemma 2.4.16 we have $\mathscr{R}_2 \otimes k(b) \cong H^0(F_b, \omega_{F_b}^{\otimes 2})$ and $\omega_{F_b}^{\otimes 2}$ is generated by global sections by Lemma 2.5.2 so such an s exists. Thus $r_{\mathscr{L},\psi}$ is globally defined and we take $\sigma = r_{\mathscr{L},\psi}$. □

For surfaces of general type, the morphism to the canonical model factors through the morphism to the relative canonical model:

2.5.4 Lemma. *Let S be a smooth projective surface of general type, B a smooth projective curve, and $f\colon S \to B$ a fibration of genus $g \geq 2$. Then the morphism $\pi\colon S \to X$ to the canonical model factors as $\pi = \tau \circ \sigma$, where $\sigma\colon S \to \mathfrak{X}$ is the morphism to the relative canonical model and $\tau\colon \mathfrak{X} \to X$ is a morphism.*

Proof. For each affine open U of B we can consider the homomorphism of graded algebras

$$\bigoplus_{n=0}^{\infty} \omega_S^{\otimes n}(S) \to \bigoplus_{n=0}^{\infty} \omega_S^{\otimes n}(f^{-1}(U))$$

given by restricting the global sections to $f^{-1}(U)$. By considering their homogeneous spectra, this homomorphism induces a morphism $\mathfrak{X}_U \to X$, where $\mathfrak{X}_U$ is the inverse image of U for the morphism $\mathfrak{X} \to B$. These morphisms glue together to give a morphism $\tau\colon \mathfrak{X} \to X$. Now the identity $\pi = \tau \circ \sigma$ follows from the definitions of π and σ.

□

The above lemma immediately implies the relative canonical model has at most rational double points as singularities:

2.5.5 Corollary. *Let S be a smooth projective surface of general type, B a smooth projective curve, $f\colon S \to B$ a relatively minimal fibration of genus $g \geq 2$ and $\mathfrak{X}$ its relative canonical model. Then $\mathfrak{X}$ has at most rational double points as singularities, and the morphism $\sigma\colon S \to \mathfrak{X}$ is a minimal resolution of singularities.*

2.5.6 Remark. One can show using the work of Ogg [17], discussed in Section 3.5, that the relative canonical model $\mathfrak{X}$ of a genus 2 fibration only has rational double points as singularities, even if S has special type.

Similarly to the canonical model, the relative canonical model contracts any (-2)-curve contained in a fibre:

2.5.7 Proposition. *Let S be a smooth projective surface of general type, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$ and $\mathfrak{X}$ its relative canonical model. If C is a (-2)-curve which is contained in F_b for some $b \in B$, then its image under σ is a point.*

Proof. The proof uses the exact same ideas as Theorem 2.3.8. Let $C \subset F_b$ be a (-2)-curve. The adjunction formula shows that we must have $\deg_C(K_S|_C) = C \cdot K_S = 0$, so we see the restriction of ω_S to C is a degree 0 line bundle, so the restriction is the trivial bundle. Thus for any n and any affine open $b \in U \subset B$, any section in $\omega_S^{\otimes n}(f^{-1}(U))$ is constant on C and thus C must be mapped to a point by σ. □

We finish this section by studying the fibre $\mathfrak{X}_b$ above $b \in B$.

2.5.8 Lemma. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$ and $\mathfrak{X}$ its relative canonical model. Denote the fibre of the morphism $\mathfrak{X} \to B$ above $b \in B$ by $\mathfrak{X}_b$. Then we have an isomorphism*

$$\mathfrak{X}_b \cong \operatorname{Proj}(\mathscr{R}(b))$$

where $\mathscr{R}(b) := b^*\mathscr{R} \cong \mathscr{R}_b \otimes_{\mathscr{O}_{B,b}} k(b)$.

Proof. $\mathfrak{X}_b$ is obtained as the inverse limit

$$\mathfrak{X}_b = \varprojlim_{b \in U} \operatorname{Proj}(\mathscr{R}(U)) = \operatorname{Proj}(\varinjlim_{b \in U} \mathscr{R}(U)),$$

and since

$$\mathscr{R}(b) = \varinjlim_{b \in U} \mathscr{R}(U)$$

we get the result. □

We will now introduce the canonical algebra of a fibre, which is analogous to the canonical ring of a surface.

2.5.9 Definition. Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$. Write ω_{F_b} for the dualizing sheaf of F_b. Then the *canonical algebra of the fibre F_b* is the $\mathbb{C}$-algebra

$$\bigoplus_{m=0}^{\infty} H^0(F_b, \omega_{F_b}^{\otimes m}).$$

The algebra $\mathscr{R}(b)$ is isomorphic to the canonical algebra:

2.5.10 Corollary. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$. Then for any $b \in B$ there is an isomorphism*

$$\mathscr{R}(b) \cong \bigoplus_{m=0}^{\infty} H^0(F_b, \omega_{F_b}^{\otimes m}),$$

and the induced morphism $F_b \to \operatorname{Proj}(H^0(F_b, \omega_{F_b}^{\otimes m}))$ is the canonical morphism from 2.3.7.

Proof. By Proposition 2.4.14 we have $\omega_{S/B}|_{F_b} \cong \omega_{F_b}$, so

$$b^*\mathscr{R}_n = b^* f_* \omega_{S/B}^{\otimes n} \cong H^0(F_b, \omega_{S/B}|_{F_b}^{\otimes n}) = H^0(F_b, \omega_{F_b}^{\otimes n}),$$

which implies the statement. □

As a consequence, we find that $\sigma : S \to \mathfrak{X}$ is an isomorphism on the smooth fibres:

2.5.11 Corollary. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a fibration of genus $g \geq 2$. If $b \in B$ is such that F_b is a smooth curve, then the morphism $\sigma|_{F_b}\colon F_b \to \mathfrak{X}_b$ is an isomorphism. In particular, we see the morphism $\sigma\colon S \to \mathfrak{X}$ is birational.*

Proof. Since for a smooth fibre the canonical sheaf ω_{F_b} is ample, the statement follows. □

Chapter 3

Genus 2 fibrations

In this chapter we will focus on relatively minimal genus 2 fibrations $f\colon S \to B$. First we study smooth fibres and study their pluricanonical maps explicitly. We then show the hyperelliptic involution on a smooth fibre extends to the whole surface, which realizes the relative canonical model $\mathfrak{X}$ as a double cover of a surface $\mathcal{C}$. We then study the fibres of $\phi\colon \mathfrak{X} \to B$, and use this to show $\mathcal{C}$ has at most rational double points as singularities and compute them explicitly in terms of local data. After this we describe the ramification locus $\mathrm{Ram}(\phi)$, and describe the singularities of $\mathfrak{X}$ outside the ramification divisor R_ϕ. Finally we will use this to determine the possible fibres $F_b \subset S$ for which the divisor R_ϕ does not have any singularities on $\mathfrak{X}_b$.

We will now describe the pluricanonical maps for a smooth fibre. For a smooth projective genus 2 curve C given on an affine chart by

$$z^2 = f(x),$$

we can consider the morphisms to projective spaces induced by the tensor powers of the canonical bundle ω_C. The global sections of ω_C are spanned by $X_0 = \frac{\mathrm{d}x}{z}$ and $X_1 = \frac{x\,\mathrm{d}x}{z}$, so we see that the canonical morphism $\Phi_{\omega_C}\colon C \to \mathbb{P}^1$ given by $(x, z) \to (X_0(x, z) : X_1(x, z)) = (1 : x)$ is a double cover. This morphism is ramified at exactly 6 points, which are the $(x, z) = (\alpha, 0)$ such that $f(\alpha) = 0$. As this morphism is invariant under the action of the hyperelliptic involution ι given by $x \mapsto x, z \mapsto -z$, we see it coincides with the natural morphism $C \to C/\langle\iota\rangle$.

There are essentially no new global sections for $\omega_C^{\otimes 2}$, since they are generated by the sections X_0^2, X_0X_1 and X_1^2 so we see $\Phi_{\omega_C^{\otimes 2}}\colon C \to \mathbb{P}^2$ is just the composition of Φ_{ω_C} with the quadratic Veronese map. On the other hand, $\omega_C^{\otimes 3}$ has the genuinely new section

$$Z = \frac{(\mathrm{d}x)^3}{z^2}$$

and we have the relation

$$Z^2 = F(X_0, X_1),$$

where F is the unique homogeneous polynomial of degree 6 satisfying $f(x) = F(1,x)$. Thus we see $\Phi_{\omega_C^{\otimes 3}}\colon C \to \mathbb{P}^5$ is an embedding, and the canonical algebra of the curve is given by $\mathbb{C}[X_0, X_1, Z]/(Z^2 - F(X_0, X_1))$. In particular, for genus 2 fibrations this gives an explicit description of the smooth fibres of the relative canonical model.

3.1 The involution on a hyperelliptic fibration

For general fibrations, there may be no simple automorphisms, but hyperelliptic fibrations induce a special involution of the surface. On each smooth fibre of a hyperelliptic fibration $f\colon S \to B$ we can consider the hyperelliptic involution. Combining these involutions we get a rational involution $\iota\colon S \to S$ defined on the smooth fibres.

3.1.1 Proposition. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a relatively minimal hyperelliptic fibration. There exists an involution $\iota\colon S \to S$ of S over B which on smooth fibres is the hyperelliptic fibration.*

Proof. Because S is normal, it follows by [13, Chapter V, Lemma 5.1] that there are only finitely many fundamental points of the rational involution ι. We blow up the surface a minimal number of times in the fundamental points to get a birational morphism $\widetilde{\iota}\colon \widetilde{S} \to S$. By Proposition 2.1.6 this morphism is thus a composition

$$\widetilde{\iota}\colon \widetilde{S} = S_r \to \cdots \to S_0 = S$$

of monoidal transformations. If E_i is the exceptional curve corresponding to $\pi_i\colon S_i \to S_{i-1}$ and $C \subset S_{i-1}$ is an irreducible curve, then $C^2 = (\pi_i^* C)^2 = \tilde{C}^2 + 2k(\tilde{C} \cdot E) - k^2 = \tilde{C}^2 + k^2$, where $\tilde{C}$ is the strict transform of C with respect to π_i and k is the multiplicity of C at the point blown up by π_i. Thus if $\tilde{C}$ is an (-1)-curve in a fibre in $\widetilde{S}$, then we need to have $C^2 \geq 0$ unless $k = 0$.

Every component in a fibre needs to have strictly negative self intersection, so $C^2 \geq 0$ is not possible. Thus we must have $k = 0$, so C is an (-1)-curve in S_{i-1}. By induction we see that $C = \widetilde{\iota}(E_r)$ must either be a point or a (-1)-curve. The latter is not possible since $f\colon S \to B$ is relatively minimal, so E_r maps to a point. Thus we see the final blowup π_r was not necessary to obtain a morphism, so the blowup was not minimal which contradicts the assumption. Therefore no blowups are necessary to turn ι into a morphism, so ι is already a morphism itself. □

3.1.2 Remark. In the above proposition, the requirement that the fibration is minimal is necessary. Consider an involution $\iota\colon S \to S$ on a surface S and $p \in S$ is a point such that $\iota(p) \neq p$, and take $\widetilde{S}$ to be the blowup of S at p. Then the rational involution ι induces on $\widetilde{S}$ does not extend to a morphism. If it did, then it would need to send $\iota(p)$ to every point on the exceptional curve, which is not possible.

The involution on S induces an involution on the relative canonical model $\mathfrak{X}$, which we also denote by ι.

3.1.3 Definition. Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a hyperelliptic fibration, and write $\iota\colon \mathfrak{X} \to \mathfrak{X}$ for the morphism induced by the hyperelliptic involution on smooth fibres. Recall that in Definition 2.5.1 we introduced the relative canonical algebra $\mathscr{R} = \bigoplus_n \mathscr{R}_n$. We define $\mathscr{R}_n^+$ and $\mathscr{R}_n^-$ to be the invariant and anti-invariant subsheaves of $\mathscr{R}_n$ with respect to the action of ι, respectively. Similarly we define $\mathscr{R}^+$ and $\mathscr{R}^-$ to be the invariant and anti-invariant subsheaves of $\mathscr{R}$.

From the definition we directly see that

$$\mathscr{R}_n = \mathscr{R}_n^+ \oplus \mathscr{R}_n^-,$$

$$\mathscr{R} = \mathscr{R}^+ \oplus \mathscr{R}^-.$$

Note that $\mathscr{R}^+$ is an $\mathcal{O}_B$-algebra, while $\mathscr{R}^-$ is an $\mathscr{R}^+$-module.

3.1.4 Definition. In the same setting as the previous definition we define $\mathcal{C}$ to be the B-scheme

$$\mathcal{C} := \mathbf{Proj}_B(\mathscr{R}^+).$$

Combining this with Proposition 1.4.7 we find the following corollary:

3.1.5 Corollary. *Let S be a smooth projective surface, B a smooth projective curve and $f\colon S \to B$ a relatively minimal hyperelliptic fibration. The inclusion $\mathscr{R}^+ \to \mathscr{R}$ induces an isomorphism*

$$\mathfrak{X}/\langle\iota\rangle \cong \mathcal{C},$$

and thus a double cover

$$\phi\colon \mathfrak{X} \to \mathcal{C}.$$

Proof. The inclusion $\mathscr{R}^+ \to \mathscr{R}$ induces a morphism

$$\phi\colon \mathfrak{X} = \mathbf{Proj}_B(\mathscr{R}) \to \mathcal{C} = \mathbf{Proj}_B(\mathscr{R}^+)$$

since the irrelevant ideal of $\mathscr{R}$ is the only prime ideal containing the irrelevant ideal of $\mathscr{R}^+$. Since ι acts trivially on $\mathbf{Proj}_B(\mathscr{R}^+)$, the morphism induces an unique morphism $g\colon \mathfrak{X}/\langle\iota\rangle \to \mathcal{C}$. The morphism g is finite and an isomorphism on function fields, so since $\mathcal{C}$ is normal it is an isomorphism. □

3.2 Fibres of the relative canonical model

In this section we will study the fibres of the map $\mathfrak{X} \to B$ for a genus 2 fibration $f\colon S \to B$ based on results by Mendes Lopes. In her results, the following definition is used:

3.2.1 Definition. Let S be as smooth projective surface with canonical divisor class K_S, let B be a smooth projective curve and $f\colon S \to B$ a fibration, and $b \in B$. We say that a divisor $A \subseteq F_b$ is an *elliptic cycle* if $A^2 = -1$ and $A \cdot K_S = 1$ (so that $p_a(A) = 1$). We call A an *elliptic tail* if it is a minimal elliptic cycle.

In her PhD thesis Mendes Lopes analysed the canonical algebras of the fibres of a genus 2 fibration, which yields the following classification theorem [15, Theorem (3.1) and Theorem (3.7)]:

3.2.2 Theorem. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration. Then the canonical algebra of the fibre F_b above $b \in B$ can be presented as*

$$\mathscr{R}(b) \cong \mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6),$$

where $\mathbb{C}[x_0, x_1, y, z]$ is the weighted polynomial ring with

$$\deg x_0 = \deg x_1 = 1,\ \deg y = 2,\ \deg z = 3,$$

and Q_2, Q_6 are homogeneous polynomials of degree 2 and 6 of the form

$$Q_2 = \lambda y - F_2(x_0, x_1)$$

and

$$Q_6 = z^2 - y^3 - F_6(x_0, x_1, y).$$

If $\lambda = 0$ we can furthermore write

$$Q_2 = x_0^2 - \alpha x_0 x_1,$$

$$Q_6 = z^2 - y^3 - x_1^2(\alpha_0 y^2 + \alpha_1 x_1^4) - x_0^2(\beta_0 y^2 + \beta_1 x_1^4).$$

If $\lambda = 0$, then the fibre F_b contains two distinct elliptic tails if $\alpha \neq 0$, and F_b contains a single elliptic tail of multiplicity 2 if $\alpha = 0$.

Furthermore y and z are invariant under the involution ι, while x_0 and x_1 are anti-invariant.

3.2.3 Definition. Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration. Using the classification of the above theorem, we can distinguish between three types of fibres F of the morphism $\mathfrak{X} \to B$:

(1) F is *honestly hyperelliptic*: F is a fibre with $\lambda \neq 0$,
(2) F has two elliptic tails: F is a fibre with $\lambda = 0$ and $\alpha \neq 0$,
(3) F has a double elliptic tail: F is a fibre with $\lambda = 0$ and $\alpha = 0$.

Since by Proposition 1.3.9 $\mathscr{R}^+$ and its subalgebra $\mathscr{R}^+_{\text{even}}$ of elements of even degree satisfy $\mathcal{C} = \text{Proj}(\mathscr{R}^+) \cong \text{Proj}(\mathscr{R}^+_{\text{even}})$, we get the following results on the fibres of $\mathcal{C} \to B$:

3.2.4 Proposition. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration and $b \in B$. Then if we write $u_0 = x_0^2$, $u_1 = x_0x_1$ and $u_2 = x_1^2$, then*

$$\mathscr{R}^+_{\text{even}}(b) \cong \begin{cases} \mathbb{C}[u_0, u_1, u_2]/(u_0u_2 - u_1^2) & \text{if } F_b \text{ is honestly hyperelliptic,} \\ \mathbb{C}[u_0, u_1, u_2, y]/(u_0u_2 - u_1^2, u_0 - u_1) & \text{if } F_b \text{ has two elliptic tails,} \\ \mathbb{C}[u_0, u_1, u_2, y]/(u_0, u_1^2) & \text{if } F_b \text{ has a double elliptic tail.} \end{cases}$$

In particular, we see that the fibre above $b \in B$ of the map $\mathcal{C} \to B$ is

(1) *A nonsingular conic when F_b is honestly hyperelliptic,*
(2) *Two transversally intersecting projective lines if F_b has two elliptic tails,*
(3) *A double projective line if F_b has a double elliptic tail.*

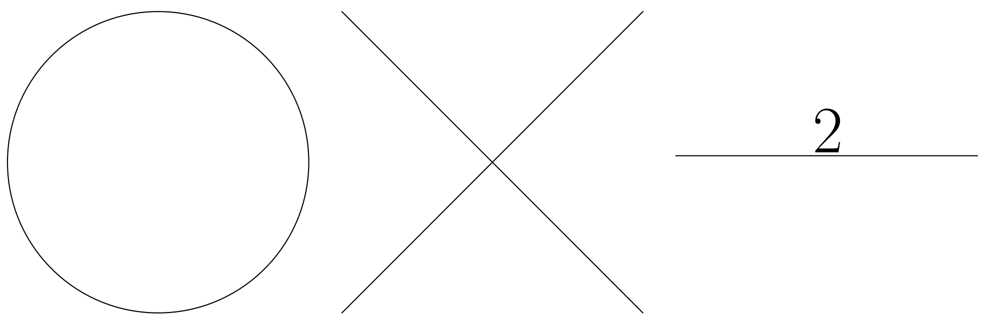


Figure 3.1: The fibre of $\mathcal{C}$ for each of the types of F_b.

Proof. Since z^2 can be expressed in terms of the other variables by the relation imposed by Q_6, we see that $\mathscr{R}^+_{\text{even}}(b) \cong \mathbb{C}[x_0^2, x_0x_1, x_1^2, y]/(Q_2)$. The result now directly follows by considering the different cases. □

Using Mendes Lopes's description, we also see that $\mathscr{R}_{\text{even}}$ is generated by $\mathscr{R}_2$ giving the following result:

3.2.5 Proposition. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration. Then there is a canonical closed immersion*

$$\mathcal{C} \to \mathbb{P}(\mathscr{R}_2)$$

into the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathscr{R}_2)$ such that $\mathcal{C}$ is a closed subspace locally given by a single quadratic equation.

Proof. By Lemma 2.4.16 we see that $\mathscr{R}_2$ is a locally free sheaf of rank 3 on B, so $\mathbb{P}(\mathscr{R}_2)$ is a $\mathbb{P}^2$-bundle on B. Since $\mathscr{R}_{\text{even}}$ is generated by $\mathscr{R}_2$ we get a surjective homomorphism $\operatorname{Sym}(\mathscr{R}_2) \to \mathscr{R}_{\text{even}}$ of graded $\mathscr{O}_B$-algebras. By [11, Proposition (2.9.2)] this morphism induces a closed immersion $\mathcal{C} \to \mathbb{P}(\mathscr{R}_2)$ of B-schemes. (The result [11, Proposition (2.9.2)] generalizes to the relative setting since the property of being a closed immersion is local in B.) Furthermore, $\mathcal{C}$ is locally given by a single quadratic equation since on fibres $b \in B$ we have $\mathscr{R}^{+}_{\text{even}}(b) \cong \mathbb{C}[x_0^2, x_0x_1, x_1^2, y]/(Q_2)$. □

By the previous proposition we directly find the following corollary:

3.2.6 Corollary. *Let S be as smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration and $b \in B$. Then the B-scheme $\mathcal{C}$ is a* conic bundle*: for any $b \in B$ the fibre of the morphism $\mathcal{C} \to B$ above b has arithmetic genus 0.*

We now give a simpler description for the fibres of the canonical model which are not honestly hyperelliptic.

3.2.7 Proposition. *Let S be a smooth projective surface, B a smooth projective curve, $f\colon S \to B$ a relatively minimal genus 2 fibration and $b \in B$. Write the canonical algebra of the fibre above $b \in B$ as*

$$\mathscr{R}(b) \cong \mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6)$$

with Q_2, Q_6 as in Theorem 3.2.2, and consider the coordinates

$$u_0 = x_0^2,\ u_1 = x_0x_1,\ u_2 = x_1^2,\ y,\ v_0 = x_0z,\ v_1 = x_1z,\ \text{and } z^2,$$

where the u_i have degree 1, the v_i have degree 2 and z^2 has degree 3. In these coordinates, we can consider the fibre $\mathfrak{X}_b$ above b as a subspace of weighted projective space. Using these coordinates, if F_b has two elliptic tails, the fibre $\mathfrak{X}_b$ is the sum of the curves

$$z^2 = \alpha_1 u_2^3 + \alpha_0 u_2 y + y^3, \quad v_1^2 = u_2 z^2, \quad u_0 = u_1 = v_0 = 0$$

and

$$z^2 = \delta_1 u_2^3 + \delta_0 u_2 y + y^3, \quad v_1^2 = u_2 z^2, \quad u_0 = u_1 = u_2, \quad v_0 = v_1.$$

Here $\delta_i := \alpha_i + \beta_i$.

If, on the other hand, F_b has a double elliptic tail, the fibre $\mathfrak{X}_b$ is twice the curve

$$z^2 = \alpha_1 u_2^3 + \alpha_0 u_2 y^2 + y^3, \quad v_1^2 = u_2 z^2, \quad u_0 = u_1 = v_0 = 0.$$

Proof. Using the coordinates we have

$$\mathscr{R}_{\text{even}}(b) = \mathbb{C}[u_0, u_1, u_2, y, v_0, v_1, z^2]/(u_0 - u_1, u_1(u_2 - u_1), v_0(v_1 - v_0), v_0^2 - u_1 z^2, v_1^2 - u_2 z^2, Q_6),$$

if F_b has two elliptic tails, while we have

$$\mathscr{R}_{\text{even}}(b) = \mathbb{C}[u_0, u_1, u_2, y, v_0, v_1, z^2]/(u_0, u_1^2, v_0^2, v_0 v_1 - u_1 z^2, v_1^2 - u_2 z^2, Q_6)$$

if F_b has a double elliptic tail. Here

$$Q_6 = z^2 - y^3 - u_2(\alpha_0 y^2 + \alpha_1 u_2^2) - u_0(\beta_0 y^2 + \beta_1 u_2^2)$$

as in Theorem 3.2.2. The result now follows by dividing out by the first three relations. □

3.3 Singularities of the conic bundle

In this section we will study the singularities of the conic bundle $\mathcal{C} = \mathbf{Proj}(\mathscr{R}^+)$ of a genus 2 fibration. The main result is the following theorem:

3.3.1 Theorem. *Let $f : S \to B$ be a relatively minimal genus 2 fibration, with relative canonical algebra $\mathscr{R}$ and conic bundle $\mathcal{C} = \mathbf{Proj}(\mathscr{R}^+)$. We write $\mathcal{C}_b$ for the fibre of $\mathcal{C} \to B$ above $b \in B$. The only singular points $\mathcal{C}$ can have are rational double points. Furthermore for any $b \in B$, we can determine precisely what types of rational double points appear in the fibre $\mathcal{C}_b$:*

- *if F_b is honestly hyperelliptic, then $\mathcal{C}_b$ does not contain any singular point of $\mathcal{C}$,*
- *if F_b has two elliptic tails, then $\mathcal{C}_b$ contains a single A_{2m-1} singularity, for an integer $m \geq 1$,*
- *if F_b has a double elliptic tail, then $\mathcal{C}_b$ either contains two A_1 singularities, or a single D_{2m} singularity, for an integer $m \geq 2$.*

3.3.2 We will carry out the proof using formal neighbourhoods of the fibre $\mathcal{C}_b$ for $b \in B$. As noted in the proof of Proposition 3.2.4 we have an isomorphism $\mathscr{R}^+_{\text{even}}(b) \cong \mathbb{C}[u_0, u_1, u_2, y]/(Q_2)$, where Q_2 is as in Theorem 3.2.2 and $u_0 = x_0^2$, $u_1 = x_0 x_1$ and $u_2 = x_1^2$. Take a uniformizer t at $b \in B$. We can consider the formal neighbourhood of around $t = 0$ on B, which gives

$$\widehat{\mathscr{R}}^+_{\text{even}} \cong \mathbb{C}[[t]][u_0, u_1, u_2, y]/(q_2(u_0, u_1, u_2, y; t), u_0 u_2 - u_1^2),$$

where q_2 is a homogeneous polynomial of degree 1 with coefficients in $\mathbb{C}[[t]]$ such that $q_2 \equiv Q_2 \bmod t$. We can therefore write

$$q_2 = \gamma_0(t)u_0 + \gamma_1(t)u_1 + \gamma_2(t)u_2 + \gamma_3(t)y,$$

with $\gamma_i(t)$ power series in t such that $q_2 \equiv Q_2 \bmod t$ holds. We split the proof up in three propositions, one for each type of fibre.

3.3.3 Proposition. *Let $f : S \to B$ be a relatively minimal genus 2 fibration, with relative canonical algebra $\mathscr{R}$ and conic bundle $\mathcal{C} = \mathbf{Proj}(\mathscr{R}^+)$. We write $\mathcal{C}_b$ for the fibre of $\mathcal{C} \to B$ above $b \in B$. Assume F_b is honestly hyperelliptic, for $b \in B$. Then $\mathcal{C}_b$ does not have any surface singularities.*

Proof. In this case we have $\gamma_3(0) \neq 0$, so by linearly changing the variable y to $y' = \frac{1}{\gamma_3(0)}(y + \sum_{i=0}^{2} \gamma_i(0)u_i)$ we can assume $Q_2 = y'$ so we can write $q_2 = \eta_0(t)u_0 + \eta_1(t)u_1 + \eta_2(t)u_2 + \eta_3(t)y'$ with $(\eta_0(t), \eta_1(t), \eta_2(t), \eta_3(t)) \equiv (0,0,0,1) \bmod t$. Thus by differentiating we see the singular points are those for which the relations $u_0 = 0$, $u_1 = 0$ and $u_2 = 0$ hold. So there are no singular points on the fibre. □

3.3.4 Proposition. *Let $f : S \to B$ be a relatively minimal genus 2 fibration, with relative canonical algebra $\mathscr{R}$ and conic bundle $\mathcal{C} = \mathbf{Proj}(\mathscr{R}^+)$. We write $\mathcal{C}_b$ for the fibre of $\mathcal{C} \to B$ above $b \in B$. Assume F_b contains two elliptic tails, for $b \in B$ and let $\gamma_3(t)$ be as in Section 3.3.2. Then $\mathcal{C}_b$ has an unique singularity, which is given in coordinates as $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1)$, which is a A_{2m-1}-singularity, where $m = \mathrm{val}_t(\gamma_3(t))$.*

Proof. In this case we have

$$(\gamma_0(t), \gamma_1(t), \gamma_2(t), \gamma_3(t)) \equiv (\alpha, -1, 0, 0) \bmod t,$$

and by a linear change of variables we can assume $\alpha = 1$. Since $\gamma_0(t)$ is a unit, we can multiply everything by $-1/\gamma_0(t)$ to find that $q_2 = 0$ is equivalent to $u_0 = \delta_1(t)u_1 + \delta_2(t)u_2 + \delta_3(t)y$ where $\delta_i := -\frac{\gamma_i}{\gamma_0}$, and $\delta_1(t) \equiv 1 \bmod t$ while $\delta_2(t) \equiv \delta_3(t) \equiv 0 \bmod t$. Substituting this in the relation $u_0u_2 - u_1^2 = 0$ we find the relation

$$\delta_1 u_1 u_2 + \delta_2 u_2^2 + \delta_3 u_2 y - u_1^2 = 0.$$

By differentiating the above equation we find that the singularities are the points with

$$\delta_1 u_2 - 2u_1 = 0$$

$$\delta_1 u_1 - 2\delta_2 u_2 = 0$$

$$\delta_3 u_2 = 0$$

$$\delta_1' u_1 u_2 + \delta_2' u_2^2 + \delta_3' u_2 y = 0.$$

Note that we see from this that $\gamma_3(t)$ is not identically zero, since otherwise all points with $u_1 = u_2 = 0$ are singular. So $t = 0$ gives

$$u_2 - 2u_1 = 0,$$

$$u_1 = 0,$$

and

$$\delta_1' u_1 u_2 + \delta_2' u_2^2 + \delta_3' u_2 y = 0,$$

where we write $\delta_i' = \delta_i'(0)$. So this only gives $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1)$ as a singularity, which allows us to restrict ourselves to the affine chart $y = 1$. Take $v_1 := u_1 - \frac{1}{2}\delta_1 u_2$ and $c(t) := \frac{1}{4}\delta_1(t)^2 - \delta_2(t)$ to simplify the relation to

$$v_1^2 + cu_2^2 - \delta_3 u_2.$$

Since $c(t) \equiv \frac{1}{4} \bmod t$ we see it is a unit, so we can define $v_2 = u_2 + \frac{\delta_3}{2c}$ to get

$$v_1^2 - cv_2^2 + \frac{\delta_3^2}{4c} = 0.$$

Now if δ_3 has valuation m, then we can write $v_1^2 - cv_2^2 - \frac{\delta_3^2}{4ct^{2m}} t^{2m}$, and since c and $\frac{\delta_3^2}{4ct^{2m}}$ are nonzero modulo t, they have a square root in $\mathbb{C}[[t]]$. So by a linear change of variables we can rewrite the equation as $w_1^2 - w_2^2 - w_3^{2m} = 0$ and thus we see that the singularity is the rational double point A_{2m-1}. □

Now we will consider fibres with a double elliptic tail.

3.3.5 Proposition. *Let $f : S \to B$ be a relatively minimal genus 2 fibration, with relative canonical algebra $\mathscr{R}$ and conic bundle $\mathcal{C} = \mathbf{Proj}(\mathscr{R}^+)$. We write $\mathcal{C}_b$ for the fibre of $\mathcal{C} \to B$ above $b \in B$. Assume F_b contains a double elliptic tail, for $b \in B$ and let $\gamma_i(t)$ be as in Section 3.3.2. We write $\gamma_i' := \gamma_i'(0)$. Then $\mathcal{C}_b$ has either one or two singularities, depending on whether $\gamma_3' = 0$. If $\gamma_3' \neq 0$ then the singularities are two A_1-singularities given as $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1)$ and $(u_0 : u_1 : u_2 : y) = (0 : 0 : -\gamma_3' : \gamma_2')$. If on the other hand $\gamma_3' = 0$, then there is only a single singularity which is given in coordinates as $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1)$, which is a D_{2m}-singularity where $m = \mathrm{val}_t(\gamma_3(t)) \geq 2$.*

Proof. In this case we have

$$(\gamma_0(t), \gamma_1(t), \gamma_2(t), \gamma_3(t)) \equiv (1, 0, 0, 0) \bmod t.$$

We proceed similarly to the previous proof. Since $\gamma_0(t)$ is a unit, we can multiply everything by $-1/\gamma_0(t)$ to find that $q_2 = 0$ is equivalent to $u_0 = \delta_1(t)u_1 + \delta_2(t)u_2 + \delta_3(t)y$

where $\delta_i := -\frac{\gamma_i}{\gamma_0}$. Note that $\delta_1(t) \equiv 1 \mod t$ and the other two are equivalent to 0. Substituting this in the relation $u_0u_2 - u_1^2 = 0$ we find the relation

$$\delta_1 u_1 u_2 + \delta_2 u_2^2 + \delta_3 u_2 y - u_1^2 = 0.$$

By differentiating the above equation we find that the singularities are the points with These are the points with

$$\delta_1 u_2 - 2u_1 = 0$$

$$\delta_1 u_1 - 2\delta_2 u_2 = 0$$

$$\delta_3 u_2 = 0$$

$$\delta_1' u_1 u_2 + \delta_2' u_2^2 + \delta_3' u_2 y = 0.$$

Note that we see from this that $\delta_3(t)$ is not identically zero, since otherwise all points with $u_1 = u_2 = 0$ are singular. For the same reason, we also see that $(\delta_2', \delta_3') \neq (0, 0)$ since otherwise all points with $u_1 = t = 0$ are singular. So $t = 0$ gives

$$u_1 = 0,$$

and

$$u_2(\delta_2' u_2 + \delta_3' y) = 0,$$

where we write $\delta_i' = \delta_i'(0)$. This gives the singular points $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1)$ and $(u_0 : u_1 : u_2 : y) = (0 : 0 : -\delta_3' : \delta_2') = (0 : 0 : -\gamma_3' : \gamma_2')$, which coincide exactly when $\delta_3' = 0$. By taking $v_1 := u_1 - \frac{1}{2}\delta_1 u_2$ and $c(t) := \frac{1}{4}\delta_1(t)^2 - \delta_2(t)$ again, we simplify the relation to

$$v_1^2 + cu_2^2 - \delta_3 u_2.$$

First we consider the case $\delta_3' \neq 0$, so that δ_3/t is a unit. We will start by studying the singularity at $(v_1 : u_2 : y) = (0 : 0 : 1)$, so we use the affine chart given by $u_3 = 1$. Define $v_2 = \frac{\delta_3}{t} u_2$ so we get

$$v_1^2 = tv_2(1 + \varepsilon v_2),$$

where $\varepsilon := \frac{tc}{\delta_3^2}$. In a formal neighbourhood of $v_2 = 0$ we see that $1 + \varepsilon(t)v_2$ is a unit. So we see that the singularity is analytically isomorphic to

$$v_1^2 = tv_2$$

and thus the singularity is an A_1-singularity.

Now we consider the remaining singularity $(0 : -\delta_3' : \delta_2')$. Take the affine chart given by $u_2 = 1$, so that the singularity is at $(0, \alpha)$ where $\alpha = -\frac{\delta_2'}{\delta_3'}$. We shift the argument u_3 by α so the equation becomes

$$v_1^2 = c(t) + \delta_3(t)(u_3 + \alpha)$$

and the studied singularity is at $(v_1, u_3) = (0, 0)$. Factoring out t we get

$$v_1^2 = t(c(t)/t + \delta_3(t)/t(u_3 + \alpha)).$$

Using the change of variables $v_3 = c(t)/t + \delta_3(t)/t(u_3 + \alpha)$ we get

$$v_1^2 = tv_3$$

so we see it is an A_1-singularity.

It remains to consider the case $\delta_3' = 0$. As remarked earlier, this implies $\delta_2' \neq 0$. Then $c(t)/t$ is an unit. Now let $\varepsilon(t)$ be a square root of $c(t)/t$ so that $\varepsilon(t)^2 = c(t)/t$. If we define $v_2 = \varepsilon u_2 + \frac{\delta_3/t}{2\varepsilon}$ then the equation becomes

$$v_1^2 = tv_2^2 + \zeta(t)$$

where $\zeta(t) = \frac{\delta_3^2}{4c}$. We therefore see the singularity is a D_{2m}-singularity where m is the valuation of δ_3 at $t = 0$ (which is at least 2). □

3.4 Singularities of the relative canonical model

In this section, we will study the singular points of $\mathfrak{X}$. The following proposition shows those singular points only come from ramification or from singularities of the conic bundle $\mathcal{C}$.

3.4.1 Proposition. *Let $f : S \to B$ be a genus 2 fibration with relative canonical model $\mathfrak{X}$ and conic bundle $\mathcal{C}$, let $\phi\colon \mathfrak{X} \to \mathcal{C}$ be the morphism from 3.1.5, and let $p \in \mathfrak{X}$. If p does not lie on the singular locus of the ramification divisor R_ϕ, defined in 1.4.11, and $\phi(p) \in \mathcal{C}$ is a smooth point, then p is a smooth point.*

Proof. Since p gets mapped to a smooth point in $\mathcal{C}$, we can consider an smooth open subvariety $U \subseteq \mathcal{C}$ containing $\phi(x)$. Furthermore, since the singular locus of the ramification divisor R_ϕ is finite, we can choose $U \subseteq \mathcal{C}$ such that $V = \phi^{-1}(U) \subseteq \mathfrak{X}$ is disjoint from this locus. Write $\phi'\colon V \to U$ for the restriction of ϕ to V, so ϕ' is a double cover of U with the ramification divisor $R_{\phi'}$ smooth. Now Lemma 1.4.14 shows V is smooth, and therefore p is a smooth point. □

In the remainder of the section, we study the points of $\mathfrak{X}$ which get mapped to singular points by ϕ. For this we first give a description of the ramification locus and branch locus of the canonical morphism

$$\phi\colon \mathfrak{X} \to \mathcal{C}.$$

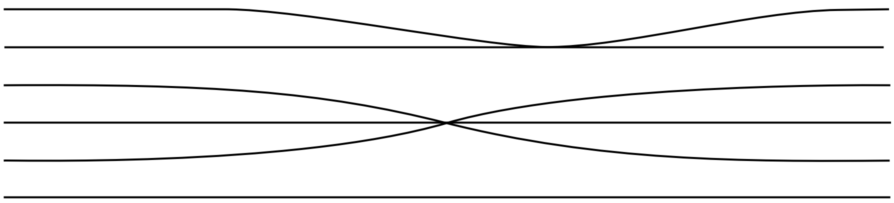


Figure 3.2: The six components of the branch divisor can intersect in singular fibres

3.4.2 Lemma. *Let $f : S \to B$ be a genus 2 fibration with relative canonical model $\mathfrak{X}$ and conic bundle $\mathcal{C}$, and let $\phi\colon \mathfrak{X} \to \mathcal{C}$ be the morphism from 3.1.5. Write*

$$\mathscr{R}(b) \cong \mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6),$$

for the canonical algebra of a fibre $b \in B$, as in Theorem 3.2.2, so $\mathrm{Proj}(\mathscr{R}(b)) \cong \mathfrak{X}_b$ *and* $\mathrm{Proj}(\mathscr{R}^+(b)) \cong \mathcal{C}_b$*. Then the restriction of the ramification divisor R_ϕ is given on $\mathfrak{X}_b$ by $z = 0$, while the branch divisor* Br_f *is given on $\mathcal{C}_b$ by $G_6 = 0$ where $G_6 := z^2 - Q_6$.*

Proof. On smooth fibres it is clear the ramification divisor is given by $z = 0$, as the restriction of ϕ to a smooth fibre is simply the bicanonical map. Let t be a uniformizer for $b \in B$. Because f is a flat morphism, there is a neighbourhood U of $\mathfrak{X}_b$ isomorphic to the subspace subspace $q_2(x_0, x_1, y : t) = 0$ and $q_6(x_0, x_1, y, z : t) = 0$ of $\mathbb{P}(1, 1, 2, 3) \times B_0$ with B_0 an affine curve, while we also have a neighbourhood $\phi(U)$ of $\mathcal{C}_b$ given as $q_2(x_0, x_1, y : t) = 0$ in $\mathbb{P}(1, 1, 2) \times B_0$. Here q_2 and q_6 are homogeneous polynomials of degree 2 and 6 in x_0, x_1, y and z with coefficients in $\mathbb{C}(t)$, such that they reduce to Q_2 and Q_6 modulo t and q_6 can be chosen such that $q_6(x_0, x_1, y, z : t) = z^2 - g_6(x_0, x_1, y : t)$ with g_6 like q_6 homogeneous of degree 6. Note that $x_0 = x_1 = y = 0$ implies $z = 0$. If we consider the affine chart $V \subset U$ given by $x_0 = 1$, we get $V \cong \mathrm{Proj}(\mathbb{C}[x_1, y, z] \times B_0/(q_2(1, x_1, y; t), z^2 - g_6(1, x_1, y : t)))$ and $\phi(V) \cong \mathrm{Proj}(\mathbb{C}[x_1, y, z] \times B_0/(q_2(1, x_1, y; t)))$. By this we see the ramification divisor on this chart is simply given by $z = 0$ and the branch divisor is given by $g_6(x_0, x_1, y : t) = 0$. We can do the same for the chart $x_1 = 1$ to find the same conclusion on that chart. There are only finitely many points $((x_0 : x_1 : y : z), t) \in U$ with $x_0 = x_1 = 0$ and each such point corresponds to a singularity of $\mathcal{C}$, as we saw in Section 3.3, proving the statement. □

This proposition shows the ramification divisor of ϕ consists of 6 branches, which only intersect in singular fibres.

For each of the singular points on the conic bundle $\mathcal{C}$ we can ask whether they lie on the branch locus of the double cover $\phi\colon \mathfrak{X} \to \mathcal{C}$, e.g. whether their inverse image consists of a single point. The following proposition shows that this is usually the case.

3.4.3 Proposition. *Let $f : S \to B$ be a genus 2 fibration with relative canonical model $\mathfrak{X}$ and conic bundle $\mathcal{C}$ and with $\phi\colon \mathfrak{X} \to \mathcal{C}$ the morphism from 3.1.5. Furthermore, let*

$b \in B$ with

$$\mathscr{R}(b) \cong \mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6)$$

with Q_2, Q_6 as in Theorem 3.2.2. If $p \in \mathfrak{X}_b$ is a point mapped to a singular point $q = \phi(p) \in \mathcal{C}_b$ then p is an isolated point in the ramification locus $\mathrm{Ram}(\phi)$*,* unless *the following conditions are* all *satisfied:*

- *the fibre F_b has a double elliptic tail,*
- *the point q is the singularity $(u_0 : u_1 : u_2 : y) = (0 : 0 : -\gamma_3' : \gamma_2')$ as given in Proposition 3.3.5 with $\gamma_3' \neq 0$.*

Furthermore, if the above conditions are satisfied then if ϕ is ramified at p, then p is in the ramification divisor R_ϕ.

Proof. As F_b does not contain singularities of S if it is honestly hyperelliptic, we can assume F_b is not honestly hyperelliptic. In the notation of Proposition 3.2.7, we see using that proposition that the unique point of $\mathfrak{X}_b$ mapped to the singularity $(u_0 : u_1 : u_2 : y) = (0 : 0 : 0 : 1) \in \mathcal{C}_b$ is $(u_0 : u_1 : u_2 : y : v_0 : v_1 : z^2) = (0 : 0 : 0 : 1 : 0 : 0 : 1)$. If F_b has two elliptic tails, then that is the only singularity of $\mathcal{C}_b$, while if F_b contains a double elliptic tail we are still left with the point $q = (u_0 : u_1 : u_2 : y) = (0 : 0 : \gamma_3' : \gamma_2')$. If $\gamma_3' = 0$ then this is the same point as what we just considered, so we assume $\gamma_3' \neq 0$. The points sent to q are the points

$$(u_0 : u_1 : u_2 : y : v_0 : v_1 : z^2) = (0 : 0 : \gamma_3' : \gamma_2' : 0 : \pm v_1 : z^2) \in \mathfrak{X}_b$$

where $v_1^2 = \gamma_3' z^2$ and

$$z^2 = \alpha_1(\gamma_3')^3 + \alpha_0\gamma_3'(\gamma_2')^2 + (\gamma_2')^3.$$

These two points differ if and only if $z^2 \neq 0$, proving the statement. □

Using the above proposition and double covers of rational points, we can derive the following theorem:

3.4.4 Theorem. *Let $f : S \to B$ be a genus 2 fibration with relative canonical model $\mathfrak{X}$ and conic bundle $\mathcal{C}$, and let S be of general type. Take $\phi\colon \mathfrak{X} \to \mathcal{C}$ to be the morphism from 3.1.5, and let $b \in B$. Consider the set*

$$A = \{x \in \mathfrak{X}_b \mid \phi(x) \in \mathcal{C}_b \text{ is a singular point}\}.$$

Then we know the cardinality of A and properties of its elements. The set A

(1) *is empty if F_b is honestly hyperelliptic,*
(2) *contains a single A_{m-1} singularity if F_b contains two elliptic tails, where m is the integer found in the proof of Theorem 3.3.1.*

If, on the other hand, F_b contains a double elliptic tail, then there are three possibilities:

(1) *if $\mathcal{C}_b$ contains a single D_{2m} singularity, then A consists of either a single D_{m+1} or a single A_{4m+5} singularity,*
(2) *if $\mathcal{C}_b$ contains two A_1 singularities and A is disjoint from the ramification divisor R_ϕ, then A consists of a smooth point and two A_1 singularities,*
(3) *if $\mathcal{C}_b$ contains two A_1 singularities and A is not disjoint from the ramification divisor R_ϕ, then A consists of a smooth point and another point which is either a rational double point or a smooth point.*

Proof. This directly follows from combining Theorem 3.3.1, Proposition 3.4.3 and Proposition 2.2.10. □

3.5 Classification of fibres

In [17], Ogg showed that every reducible fibre F_b of a relatively minimal genus 2 fibration $f\colon S \to B$ is of one of 44 types. (However, he missed 3 variants of type 41, which are described in [16].) He proved that a reducible fibre can be written as $F_b = \sum_i n_i\Gamma_i$ for integers n_i, and there are 5 different possible types a component $\Gamma := \Gamma_i$ of such a fibre can have:

$$\begin{aligned}
&\text{type A}: \quad \Gamma \cdot K_S = 1, \quad \Gamma^2 = -1, \quad p_a(\Gamma) = 1,\\
&\text{type B}: \quad \Gamma \cdot K_S = 1, \quad \Gamma^2 = -3, \quad p_a(\Gamma) = 0,\\
&\text{type C}: \quad \Gamma \cdot K_S = 2, \quad \Gamma^2 = -2, \quad p_a(\Gamma) = 1,\\
&\text{type D}: \quad \Gamma \cdot K_S = 2, \quad \Gamma^2 = -4, \quad p_a(\Gamma) = 0,\\
&\text{type E}: \quad \Gamma \cdot K_S = 0, \quad \Gamma^2 = -2, \quad p_a(\Gamma) = 0,
\end{aligned}$$

where we note that a component has type E exactly when it is a (-2)-curve. Since by the adjunction formula we must have $\sum_i n_i(\Gamma_i \cdot K_S) = 2$, we see a reducible fibre contains either one component of multiplicity one of type C or D, or two components of type A or B, or one component of multiplicity two of type A or B; while all other components are of type E. The morphism $\sigma|_{F_b}\colon F_b \to \mathfrak{X}_b$ contracts curves of type E to a point. Furthermore if S is of general type, then $\mathfrak{X}$ has only rational double points so other components are not contracted by Proposition 2.2.5. Therefore we can determine from the components what kind of canonical fibre it has:

3.5.1 Proposition. *Let S be a smooth projective surface of general type, B be a smooth projective curve and $f : S \to B$ be a relatively minimal genus 2 fibration and let $b \in B$. Then*

(1) *F_b is honestly hyperelliptic iff it contains a component of type* C *or* D*,*

(2) *F_b has two elliptic tails iff it contains two components of type* A *or* B*,*
(3) *F_b has a double elliptic tail iff it contains a component of multiplicity two of type* A *or* B*.*

Furthermore, the surface singularities which lie in the canonical fibre $\mathfrak{X}$ can be read off from the Dynkin diagrams of the cycles of (-2)-curves in F_b.

While Oggs list shows that every type of rational double point can occur on a canonical fibre, we have already seen in Theorem 3.4.4 that if R_ϕ has no singularities on $\mathfrak{X}_b$, then not all singularities can occur. The next theorem determines which fibres from Oggs classification can occur for such types.

3.5.2 Theorem. *Let S be a smooth projective surface of general type, B a smooth projective curve, $f\colon S \to B$ a relatively minimal fibration of genus 2. Let $\mathfrak{X}$ and $\mathcal{C}$ be the relative canonical model and the conic bundle of the fibration, and let $\phi\colon \mathfrak{X} \to \mathcal{C}$ be the morphism from 3.1.5. If the ramification divisor R_ϕ does not have any singularities on $\mathfrak{X}_b$, then $\mathfrak{X}_b$ is of one of the following types of Oggs classification:*

- *If F_b is honestly hyperelliptic, then it is an irreducible curve of arithmetic genus 2.*
- *If F_b has two elliptic tails, then it is a fibre of type 13, 34, 35, 36, 42(a) or 43.*
- *If F_b has a double elliptic tail, then it is a fibre of type 12, 16, 17, 18, 19, 20 or 30.*

Type 42(a) here refers to type 42 with the first option chosen for the **Kod**′*-cycle.*
The description of these types of fibres can be found in the overview at the end of this chapter, and the full list of fibres can be found in [17] plus the missed fibres found in [16].

Proof. Note that since f is relatively minimal, $\sigma\colon S \to \mathfrak{X}$ is a minimal resolution of singularities. We treat each of the types of fibres separately. If F_b is honestly hyperelliptic, then Theorem 3.4.4 implies $\mathfrak{X}_b$ contains no singularities of $\mathfrak{X}$, so σ gives an isomorphism $F_b \cong \mathfrak{X}_b$.

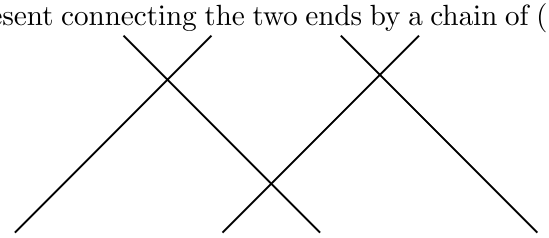
If F_b contains two elliptic tails, then $\mathfrak{X}_b$ contains a single A_n-singularity, and $\mathfrak{X}_b$ consists of two intersecting curves of arithmetic genus at most 1, which therefore need to correspond to components of type A or B. From Oggs classification, only fibres of type 13, 34, 35, 36, 42(a) and 43 contain two of these components and a single configuration of (-2)-curves corresponding to an A_n-singularity.

Now assume F_b contains a double elliptic tail, so $\mathfrak{X}_b$ is of the form $2C$ for an irreducible curve C. The curve C corresponds to a component of type A or B in F_b. If $\mathfrak{X}_b$ contains two A_1 singularities, then F_b has to have type 12 where the chain is chosen to have a length of zero, as this is the only fibre containing a component of type 2A or 2B and containing precisely two A_1 singularities.

If on the other hand $\mathfrak{X}_b$ contains a single singularity then from Oggs classification, only fibres of type type 12, 17, 18, 19, 20 or 30 contains a component of type 2A or 2B while only having a single cycle of (-2)-curves. □

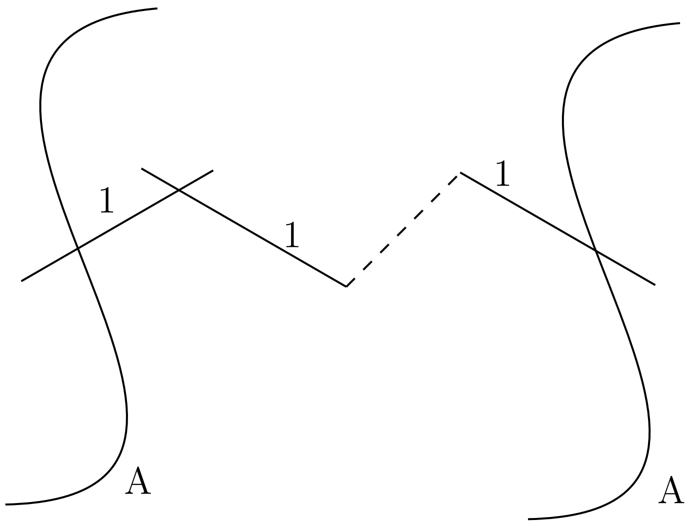
Diagrams for genus 2 fibres

In this appendix the types of fibres from Theorem 3.5.2 are given using diagrams, in the same way as Ogg originally did. The components shown in the diagram are either (-2)-curves, with multiplicity given by the nearest number, or components of type A or B. The dashed lines represent connecting the two ends by a chain of (-2)-curves of the form

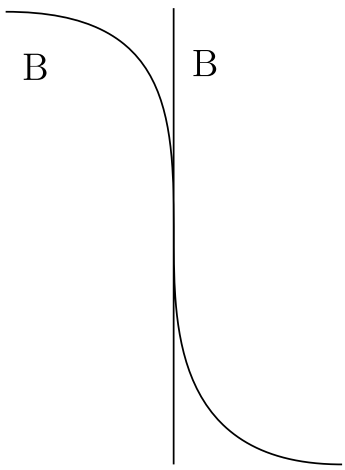


where the multiplicity of the components is given by the multiplicity of the end points. If one end of the chain is connected to a single curve, then this chain may have length zero, meaning the curve connected to one end of the chain intersect the curves on the other end. If two components intersect each other in a point with a higher multiplicity, then they intersect with multiplicity 2 unless denoted otherwise.

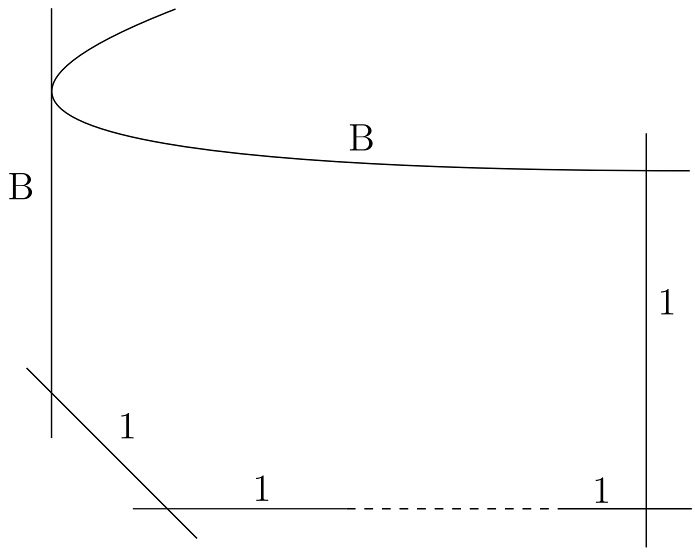
Genus 2 fibres with two elliptic tails



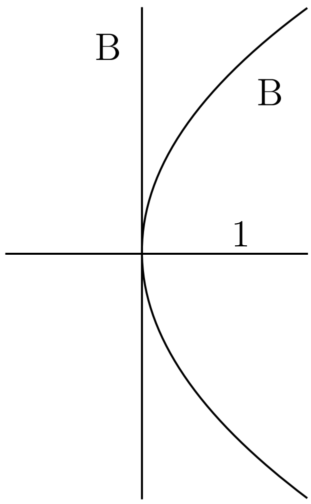
Type 13.



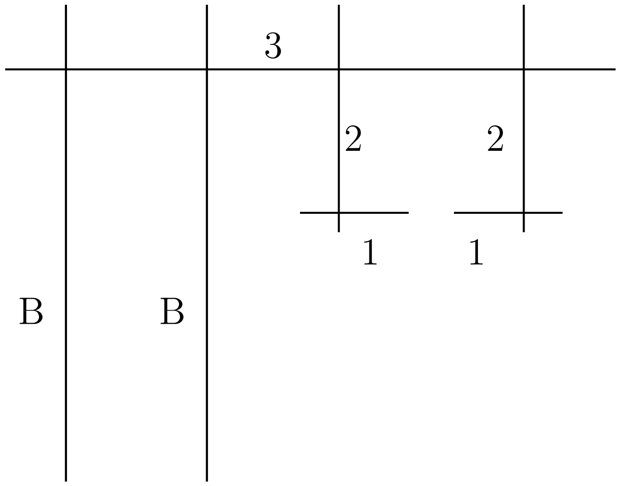
Type 34, with triple contact between the two components.



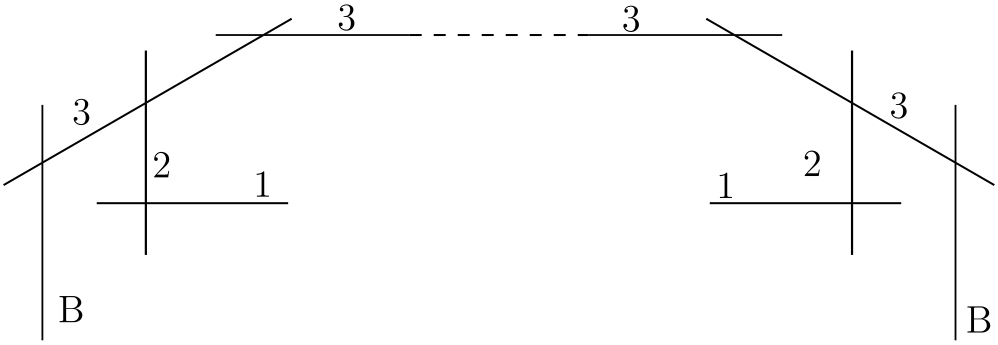
Type 35.



Type 36.

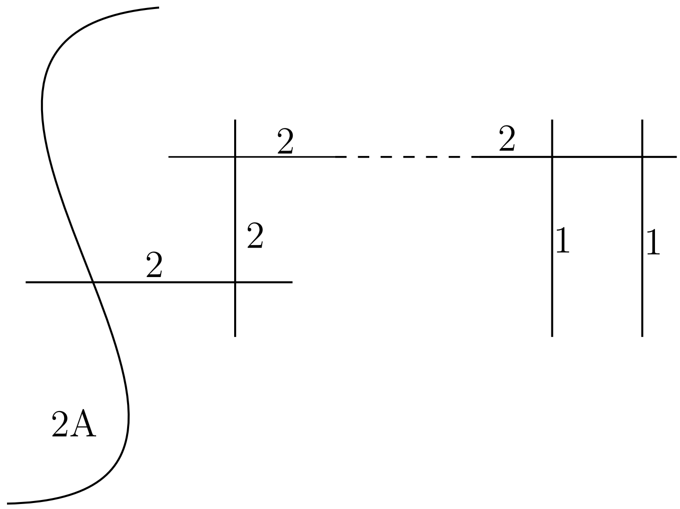


Type 42(a).

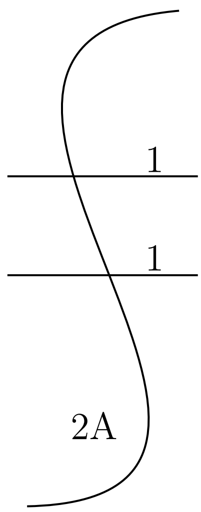


Type 43.

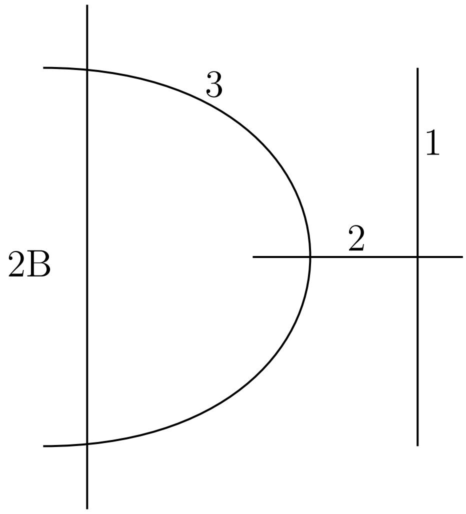
Genus 2 fibres with a double elliptic tail



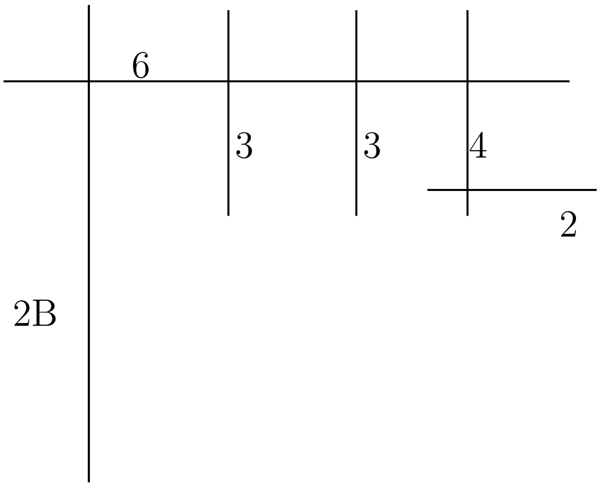
Type 12.



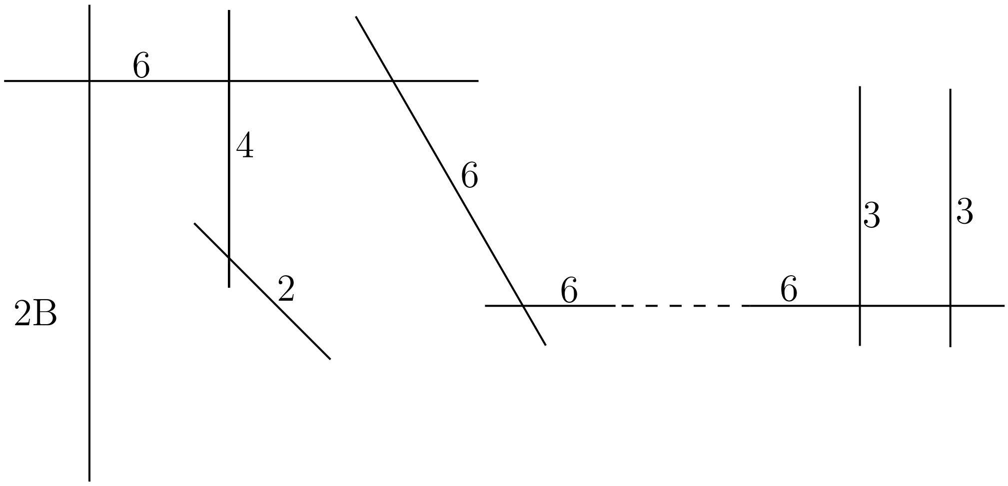
A special case of type 12, where the chain is chosen to have length 0.



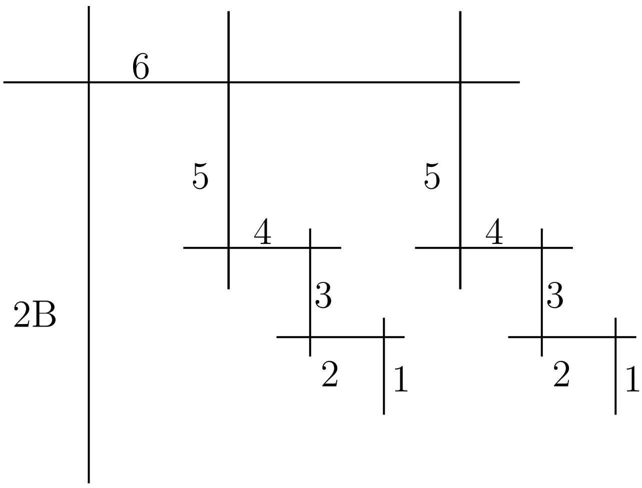
Type 16.



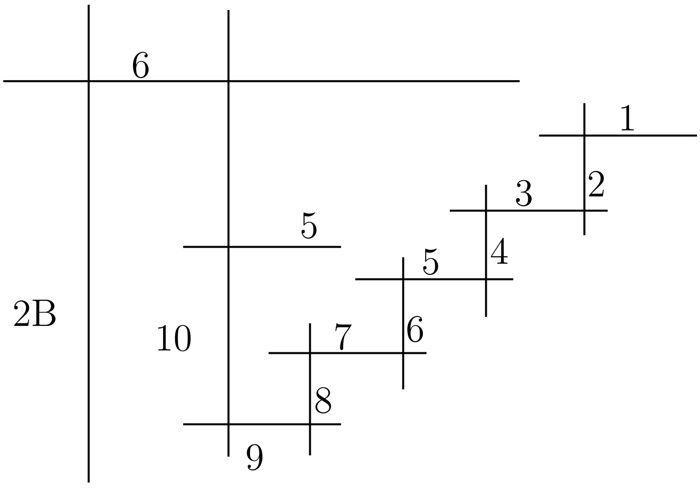
Type 17.



Type 18.



Type 19.



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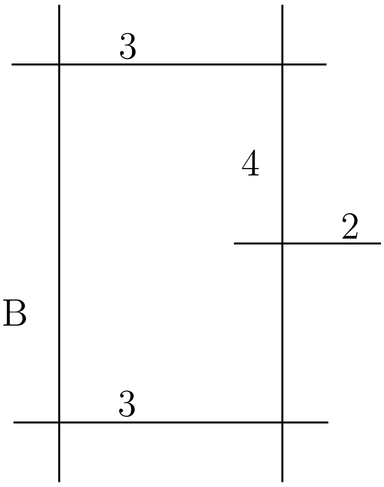

Type 30.

References

[1] Allen Altman and Steven Kleiman. *Introduction to Grothendieck Duality Theory*. Springer, Berlin, Heidelberg, 1970.

[2] Michael Artin. On isolated rational singularities of surfaces. *American Journal of Mathematics*, 88(1):129–136, 1966.

[3] Michael Artin. Coverings of the rational double points in characteristic p. In *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira*, page 11–22. Cambridge University Press, 1977.

[4] Lucian Badescu. *Algebraic Surfaces*. Graduate Texts in Mathematics. Springer, 2001.

[5] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius van de Ven. *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2015.

[6] Enrico Bombieri. Canonical models of surfaces of general type. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 42(1):171–219, Jan 1973.

[7] Egbert Brieskorn. Rationale Singularitäten komplexer Flächen. *Inventiones mathematicae*, 4(5):336–358, Oct 1968.

[8] Fabrizio Catanese and Roberto Pignatelli. Fibrations of low genus, I. *Annales Scientifiques de l'École Normale Supérieure*, 39(6):1011–1049, 2006.

[9] Alan Durfee. Fifteen characterizations of rational double points and simple critical points. *L'Enseignement Mathématique*, 25:131–163, 1979.

[10] Bas Edixhoven, Gerard van der Geer, and Ben Moonen. Quotients by group schemes. `https://www.math.ru.nl/~bmoonen/BookAV/Quotients.pdf`, 2021.

[11] Alexander Grothendieck. Éléments de géométrie algébrique : II. Étude globale élémentaire de quelques classes de morphismes. *Publications Mathématiques de l'IHÉS*, 8:5–222, 1961.

[12] Alexander Grothendieck. Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Seconde partie. *Publications Mathématiques de l'IHÉS*, 17:5–91, 1963.

[13] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate texts in mathematics*. Springer-Verlag, 1977.

[14] Kunihiko Kodaira. Pluricanonical systems on algebraic surfaces of general type. *Proceedings of the National Academy of Sciences*, 58(3):911–915, 1967.

[15] Margarida Mendes Lopes. *The relative canonical algebra for genus 3 fibrations*. PhD thesis, University of Warwick, 1988. A PDF version can be found at `http://wrap.warwick.ac.uk/78824/1/WRAP_Theses_Lopes_1988.pdf`.

[16] Yukihiko Namikawa and Kenji Ueno. The complete classification of fibres in pencils of curves of genus two. *manuscripta mathematica*, 9(2):143–186, Jun 1973.

[17] Andrew P. Ogg. On pencils of curves of genus two. *Topology*, 5(4):355–362, 1966.

[18] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2021.